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# Preface

The purpose of this book is to present arbitrage theory and its applications to pricing problems for financial derivatives. It is intended as a textbook for graduate and advanced undergraduate students in finance, economics, mathematics, and statistics and I also hope that it will be useful for practitioners.

Because of its intended audience, the book does not presuppose any previous knowledge of abstract measure theory. The only mathematical prerequisites are advanced calculus and a basic course in probability theory. No previous knowledge in economics or finance is assumed.

The book starts by contradicting its own title, in the sense that the second chapter is devoted to the binomial model. After that, the theory is exclusively developed in continuous time.

The main mathematical tool used in the book is the theory of stochastic differential equations (SDEs), and instead of going into the technical details concerning the foundations of that theory I have focused on *applications.* The object is to give the reader, as quickly and painlessly as possible, a solid working knowledge of the powerful mathematical tool known as Itö calculus. We treat basic SDE techniques, including Feynman-Kač representations and the Kolmogorov equations. Martingales are introduced at an early stage. Throughout the book there is a strong emphasis on concrete computations, and the exercises at the end of each chapter constitute an integral part of the text.

The mathematics developed in the first part of the book is then applied to arbitrage pricing of financial derivatives. We cover the basic Black-Scholes theory, including delta hedging and “the greeks”, and we extend it to the case of several underlying assets (including stochastic interest rates) as well as to dividend paying assets. Barrier options, as well as currency and quanto products, are given separate chapters. We also consider, in some detail, incomplete markets.

American contracts are treated only in passing. The reason for this is that the theory is complicated and that few analytical results are available. Instead I have included a chapter on stochastic optimal control and its applications to optimal portfolio selection.

Interest rate theory constitutes a large part of the book, and we cover the basic short rate theory, including inversion of the yield curve and affine term structures. The Heath-Jarrow-Morton theory is treated, both under the objective measure and under a martingale measure, and we also present the Musiela parametrization. The basic framework for most chapters is that of a multifactor model, and this allows us, despite the fact that we do not formally use measure theory, to give a fairly complete treatment of the general change of numeraire technique which is so essential to modern interest rate theory. In particular we treat forward neutral measures in some detail. This allows us to present the Geman-El Karoui-Rochet formula for option pricing, and we apply it to the general Gaussian forward rate model, as well as to a number of particular cases.

Concerning the mathematical level, the book falls between the elementary text by Hull (1997), and more advanced texts such as Duffie (1996) or Musiela and Rutkowski (1997). These books are used as canonical references in the present text.

In order to facilitate using the book for shorter courses, the pedagogical approach has been that of first presenting and analyzing a simple (typically one-dimensional) model, and then to derive the theory in a more complicated (multidimensional) framework. The drawback of this approach is of course that some arguments are being repeated, but this seems to be unavoidable, and I can only apologize to the technically more advanced reader.

Notes to the literature can be found at the end of most chapters. I have tried to keep the reference list on a manageable scale, but any serious omission is unintentional, and I will be happy to correct it. For more bibliographic information the reader is referred to Duffie (1996) and to Musiela and Rutkowski (1997) which both contain encyclopedic bibliographies.

On the more technical side the following facts can be mentioned. I have tried to present a reasonably honest picture of SDE theory, including Feynman-Kač representations, while avoiding the explicit use of abstract measure theory. Because of the chosen technical level, the arguments concerning the construction of the stochastic integral are thus forced to be more or less heuristic. Nevertheless I have tried to be as precise as possible, so even the heuristic arguments are the "correct" ones in the sense that they can be completed to formal proofs. In the rest of the text I try give full proofs of all mathematical statements, with the exception that I have often left out the checking of various integrability conditions.

Since the Girsanov theory for absolutely continuous changes of measures is outside the scope of this text, martingale measures are introduced by the use of locally riskless portfolios, partial differential equations (PDEs) and the Feynman-Kač representation theorem. Still, the approach to arbitrage theory presented in the text is basically a probabilistic one, emphasizing the use of martingale measures for the computation of prices.

The integral representation theorem for martingales adapted to a Wiener filtration is also outside the scope of the book. Thus we do not treat market completeness in full generality, but restrict ourselves to a Markovian framework. For most applications this is, however, general enough.

# Acknowledgements:

Bertil Näslund, Staffan Viotti, Peter Jennergren and Ragnar Lindgren persuaded me to start studying financial economics, and they have constantly and generously shared their knowledge with me.

Hans Bühlman, Paul Embrechts and Hans Gerber gave me the opportunity to give a series of lectures for a summer school at Monte Verita in Ascona 1995. This summer school was for me an extremely happy and fruitful time, as well as the start of a partially new career. The set of lecture notes produced for that occasion is the basis for the present book.

Over the years of writing, I have received valuable comments and advice from large number of people. My greatest debt is to Camilla Landén who has given me more good advice (and pointed out more errors) than I thought was humanly possible. I am also highly indebted to Flavio Angelini, Pia Berg, Nick Bingham, Samuel Cox, Darrell Duffie, Otto Elmgart, Malin Engström, Jan Ericsson, Damir Filipović, Andrea Gombani, Stefano Herzel, David Lando, Angus MacDonald, Alexander Matros, Ragnar Norberg, Joel Reneby, Wolfgang Runggaldier, Per Sjöberg, Patrik Säfvenblad, Nick Webber and Anna Vorwerk.

The main part of this book has been written while I have been at the Finance Department of the Stockholm School of Economics. I am deeply indebted to the school, the department and the staff working there for support and encouragement.

Parts of the book were written while I was still at the mathematics department of KTH, Stockholm. It is a pleasure to acknowledge the support I got from the department and from the persons within it.

Finally I would like to express my deeply felt gratitude to Andrew Schuller, James Martin and Kim Roberts, all at Oxford University Press, and Neville Hankins, the freelance copy-editor who worked on the book. The help given (and patience shown) by these people has been remarkable and invaluable.

Stockholm, July 1998.

Tomas Björk

# Contents

# 1 Introduction

## 1.1 Problem Formulation

The main project in this book consists in studying theoretical pricing models for those financial assets which are known as **financial derivatives** . Before we give the formal definition of the concept of a financial derivative we will, however, by means of a concrete example, introduce the single most important example: the European call option.

Let us thus consider the Swedish company *C&H*, which today (denoted by $t$=0) has signed a contract with an American counterpart *ACME*. The contract stipulates that *ACME* will deliver 1000 computer games to *C&H* exactly six months from now (denoted by $t= T$). Furthermore it is stipulated that *C&H* will pay 1000 US dollars per game to *ACME* at the time of delivery (i.e. at $t= T$). For the sake of argument we assume that the present spot currency rate between the Swedish krona (SEK) and the US dollar is 8.00 SEK/$.

One of the problems with this contract from the point of view of *C&H* is that it involves a considerable **currency risk** . Since *C&H* does not know the currency rate prevailing six months from now, this means that it does not know how many SEK it will have to pay at $t= T$. If the currency rate at $t= T$ is still 8.00 SEK/$ it will have to pay 8,000,000 SEK, but if the rate rises to, say, 8.50 it will face a cost of 8,500,000 SEK. Thus *C&H* faces the problem of how to guard itself against this currency risk, and we now list a number of natural strategies.

1. The most naive stratgey for *C&H* is perhaps that of buying one $1,000,000 **today** at the price of 8,000,000 SEK, and then keeping this money (in a Eurodollar account) for six months. The advantage of this procedure is of course that the currency risk is completely eliminated, but there are also some drawbacks. First of all the strategy above has the consequence of tying up a substantial amount of money for a long period of time, but an even more serious objection may be that *C&H* perhaps does not have access to 8,000,000 SEK today.
2. A more sophisticated arrangement, which does not require any outlays at all today, is that *C&H* goes to the forward market and buys a **forward contract** for $1,000,000 with delivery six months from now. Such a contract may, for example, be negotiated with a commercial bank, and in the contract two things will be stipulated.

    - The bank will, at $t= T$, deliver $1,000,000 to *C&H*.

- *C&H* will, at $t = T$, pay for this delivery at the rate of $K$ SEK/$. The exchange rate $K$, which is called the **forward price** , (or forward exchange rate) at $t$=0, for delivery at $t = T$, is determined at $t$=0. By the definition of a forward contract, the cost of entering the contract equals zero, and the forward rate $K$ is thus determined by supply and demand on the forward market. Observe, however, that even if the price of entering the forward contract (at $t$=0) is zero, the contract may very well fetch a nonzero price during the interval $[0, T]$.

Let us now assume that the forward rate today for delivery in six months equals 8.10 SEK/$. If *C&H* enters the forward contract this simply means that there are no outlays today, and that in six months it will get one $1,000,000 at the predetermined total price of 8,100,000 SEK. Since the forward rate is determined today, *C&H* has again completely eliminated the currency risk.

However, the forward contract also has some drawbacks, which are related to the fact that a forward contract is a **binding** contract. To see this let us look at two scenarios.

- Suppose that the spot currency rate at $t = T$ turns out to be 8.20. Then *C&H* can congratulate itself, because it can now buy dollars at the rate 8.10 despite the fact that the market rate is 8.20. In terms of the million dollars at stake *C&H* has thereby made an indirect profit of 8,200,000 $[0,T]$ 8,100,000 = 100,000 SEK.
- Suppose on the other hand that the spot exchange rate at $t = T$ turns out to be 7.90. Because of the forward contract this means that *C&H* is forced to buy dollars at the rate of 8.10 despite the fact that the market rate is 7.90, which implies an indirect loss of 8,100,000 $[0,T]$ 7,900,000 = 200,000 SEK.

3. What *C&H* would like to have of course is a contract which guards it against a high spot rate at $t = T$, while still allowing it to take advantage of a low spot rate at $t = T$. Such contracts do in fact exist, and they are called **European call options** . We will now go on to give a formal definition of such an option.

**Definition 1.1** *A* **European call option** *on the amount of* $X$ *US dollars, with* **strike price** *(exercise price)* $K$ *SEK/$ and* **exercise date** $T$ *is a contract written at* $t$=0 *with the following properties.*

- *The holder of the contract has, exactly at the time* $t = T$, *the* **right** *to buy* $X$ *US dollars at the price* $K$ *SEK/$.*
- *The holder of the option has no obligation to buy the dollars.*

Concerning the nomenclature, the contract is called an option precisely because it gives the holder the option (as opposed to the obligation) of buying some **underlying** asset (in this case US dollars). A **call** option gives the holder the right to buy, wheareas a **put** option gives the holder the right to sell the underlying object at a prespecified price. The prefix **European** means that the option can only be exercised at exactly the date of expiration. There also exist **American** options, which give the holder the right to exercise the option at any time before the date of expiration.

Options of the type above (and with many variations) are traded on options markets all over the world, and the underlying objects can be anything from foreign currencies to stocks, oranges, timber or pig stomachs. For a given underlying object there are typically a large number of options with different dates of expiration and different strike prices.

We now see that *C&H* can insure itself against the currency risk very elegantly by buying a European call option, expiring six months from now, on a million dollars with a strike price of, for example, 8.00 SEK/$. If the spot exchange rate at $T$ exceeds the strike price, say that it is 8.20, then *C & H* exercises the option and buys at 8.00 SEK/$. Should the spot exchange rate at $T$ fall below the strike price, it simply abstains from exercising the option.

Note, however, that in contrast to a forward contract, which by definition has the price zero at the time at which it is entered, an option will always have a nonnegative price, which is determined on the existing options market. This means that our friends in *C&H* will have the rather delicate problem of determining exactly which option they wish to buy, since a higher strike price (for a call option) will reduce the price of the option.

One of the main problems in this book is to see what can be said from a theoretical point of view about the market price of an option like the one above. In this context it is worth noting that the European call has some properties which turn out to be fundamental.

- Since the value of the option (at $T$) depends on the future level of the spot exchange rate, the holding of an option is equivalent to a **future stochastic claim** .
- The option is a **derivative asset** in the sense that it is **defined** in terms of some **underlying** financial asset.

Since the value of the option is contingent on the evolution of the exchange rate, the option is often called a **contingent claim** . Later on we will give a precise mathematical definition of this concept, but for the moment the informal definition above will do. An option is just one example of a financial derivative, and a far from complete list of commonly traded derivatives is given below.

- European calls and puts
- American options
- Forward rate agreements
- Convertibles
- Futures
- Bonds and bond options
- Caps and floors
- Interest rate swaps

Later on we will give precise definitions of (most of) these contracts, but at the moment the main point is the fact that financial derivatives exist in a great variety and are traded in huge volumes. We can now formulate the two main problems which concern us in the rest of the book.

**Main Problems:** Take a fixed derivative as given.

- What is a "fair" price for the contract?
- Suppose that we have sold a derivative, such as a call option. Then we have exposed ourselves to a certain amount of financial risk at the date of expiration. How do we protect ("hedge") ourselves against this risk?

Let us look more closely at the pricing question above. There exist two natural and mutually contradictory answers.

**Answer 1:** "Using standard principles of operations research, a reasonable price for the derivative is obtained by computing the expected value of the discounted future stochastic payoff."

**Answer 2:** "Using standard economic reasoning, the price of a contingent claim, like the price of any other commodity, will be determined by market forces. In particular it will be determined by the supply and demand curves for the market for derivatives. Supply and demand will in their turn be influenced by such factors as aggregate risk aversion, liquidity preferences, etc., so it is impossible to say anything concrete about the theoretical price of a derivative."

The reason that there is such a thing as a theory for derivatives lies in the following fact.

**Main Result:** *Both answers above are incorrect! It* **is** *possible (given, of course, some assumptions) to talk about the "correct" price of a derivative, and this price is* **not** *computed by the method given in Answer 1 above.*

In the succeeding chapters we will analyze these problems in detail, but we can already state the basic philosophy here. The main ideas are as follows.

**Main Ideas:**

- A financial derivative is **defined in terms of** some underlying asset which already exists on the market.
- The derivative cannot therefore be priced arbitrarily **in relation to the underlying prices** if we want to **avoid mispricing between the derivative and the underlying price** .
- We thus want to price the derivative in a way that is **consistent** with the underlying prices given by the market.
- We are **not** trying to compute the price of the derivative in some "absolute" sense. The idea instead is to determine the price of the derivative **in terms of the market prices of the underlying assets** .

# 2 The Binomial Model

In this chapter we will study, in some detail, the simplest possible nontrivial model of a financial market—the binomial model. This is a discrete time model, but despite the fact that the main purpose of the book concerns continuous time models, the binomial model is well worth studying. The model is very easy to understand, almost all important concepts which we will study later on already appear in the binomial case, the mathematics required to analyze it is at high school level, and last but not least the binomial model is often used in practice.

## 2.1 The One Period Model

We start with the one period version of the model. In the next section we will (easily) extend the model to an arbitrary number of periods.

### 2.1.1 Model Description

Running time is denoted by the letter $t$, and by definition we have two points in time, $t$=0 ("today") and $t$=1 ("tomorrow"). In the model we have two assets: a **bond** and a **stock** . At time $t$ the price of a bond is denoted by $B_t$, and the price of one share of the stock is denoted by $S_t$. Thus we have two price processes $B$ and $S$.

The bond price process is deterministic and given by

$$\begin{aligned} B_0 &= 1, \\ B_1 &= 1 + R. \end{aligned}$$

The constant $R$ is the spot rate for the period, and we can also interpret the existence of the bond as the existence of a bank with $R$ as its rate of interest.

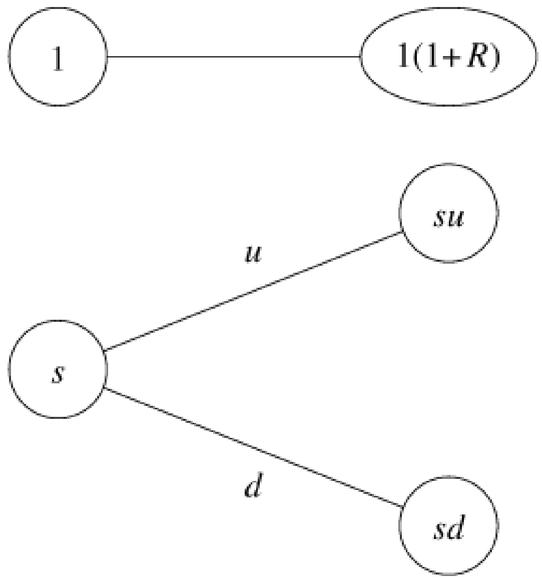
The stock price process is a stochastic process, and its dynamical behaviour is described as follows.

$$\begin{aligned} S_0 &= s, \\ S_1 &= \begin{cases} s \cdot u, & \text{with probability } p_u. \\ s \cdot d, & \text{with probability } p_d. \end{cases} \end{aligned}$$

It is often convenient to write this as

$$\begin{cases} S_0 &= s, \\ S_1 &= s \cdot Z, \end{cases}$$

Fig. 2.1 Price Dynamics




where $Z$ is a stochastic variable defined as

$$Z = \begin{cases} u, & \text{with probability } p_u. \\ d, & \text{with probability } p_d. \end{cases}$$

We assume that today's stock price $s$ is known, as are the positive constants $u$, $d$, $p_u$ and $p_d$. We assume that $d < u$, and we have of course $p_u + p_d = 1$. We can illustrate the price dynamics using the tree structure in Fig. 2.1.

## 2.1.2 Portfolios and Arbitrage

We will study the behaviour of various **portfolios** on the $(B,S)$ market, and to this end we define a portfolio as a vector $h=(x,y)$. The interpretation is that $x$ is the number of bonds we hold in our portfolio, whereas $y$ is the number of units of the stock held by us. Note that it is quite acceptable for $x$ and $y$ to be positive as well as negative. If, for example, $x = 3$, this means that we have bought three bonds at time $t$=0. If on the other hand $y=-2$, this means that we have sold two shares of the stock at time $t$=0. In financial jargon we have a **long** position in the bond and a **short** position in the stock. It is an important assumption of the model that short positions are allowed.

**Assumption 2.1.1** *We assume the following institutional facts.*

- *Short positions, as well as fractional holdings, are allowed. In mathematical terms this means that every* $h \in R^2$ *is an allowed portfolio.*
- *There is no bid-ask spread, i.e. the selling price is equal to the buying price of all assets.*
- *There are no transactions costs of trading.*

- *The market is completely liquid, i.e. it is always possible to buy and/or sell unlimited quantities on the market. In particular it is possible to borrow unlimited amounts from the bank (by selling bonds short).*

Consider now a fixed portfolio $h=(x,y)$. This portfolio has a deterministic market value at $t=0$ and a stochastic value at $t=1$.

**Definition 2.1** *The* **value process** *of the portfolio h is defined by*

$$V_t^h = xB_t + yS_t, t = 0,1,$$

*or, in more detail,*

$$\begin{aligned} V_0^h &= x + ys, \\ V_1^h &= x(1+R) + ysZ. \end{aligned}$$

Everyone wants to make a profit by trading on the market, and in this context a so called arbitrage portfolio is a dream come true; this is one of the central concepts of the theory.

**Definition 2.2** *An* **arbitrage** *portfolio is a portfolio h with the properties*

$$\begin{aligned} V_0^h &= 0, \\ V_1^h &> 0, \text{ with probability } 1. \end{aligned}$$

An arbitrage portfolio is thus basically a deterministic money making machine, and we interpret the existence of an arbitrage portfolio as equivalent to a serious case of mispricing on the market. It is now natural to investigate when a given market model is arbitrage free, i.e. when there are no arbitrage portfolios.

**Proposition 2.3** *The model above is free of arbitrage if and only if the following conditions hold.*

$$d \le (1+R) \le u. \tag{2.1}$$

**Proof** The condition (2.1) has an easy economic interpretation. It simply says that the return on the stock is not allowed to dominate the return on the bond and vice versa. To show that absence of arbitrage implies (2.1), we assume that (2.1) does in fact not hold, and then we show that this implies an arbitrage opportunity. Let us thus assume that one of the inequalities in (2.1) does not hold, so that we have, say, the inequality $s(1+R) > su$. Then we also have $s(1+R) > sd$ so it is always more profitable to invest in the bond than in the stock. An arbitrage strategy is now formed by the portfolio $h=(s,-1)$, i.e. we sell the stock short and invest all the money in the bond. For this portfolio we obviously have $V_0^h = 0$, and as for $t=1$ we have

$$V_1^h = s(1+R) - sZ,$$

which by assumption is positive.

Now assume that (2.1) is satisfied. To show that this implies absence of arbitrage let us consider an arbitrary portfolio such that $V_0^h = 0$. We thus have $x + ys = 0$, i.e. $x = -ys$. Using this relation we can write the value of the portfolio at $t=1$ as

$$V_1^h = \begin{cases} ys[u - (1+R)], & \text{if } Z = u. \\ ys[d - (1+R)], & \text{if } Z = d. \end{cases}$$

Assume now that $y>0$. Then $h$ is an arbitrage strategy if and only if we have the inequalities

$$u > 1 + R,$$
$$d > 1 + R,$$

but this is impossible because of the condition (2.1). The case $y < 0$ is treated similarly.

At first glance this result is perhaps only moderately exciting, but we may write it in a more suggestive form. To say that (2.1) holds is equivalent to saying that 1+R is a convex combination of $u$ and $d$, i.e.

$$1 + R = q_u \cdot u + q_d \cdot d,$$

where $q_u, q_d \geq 0$ and $q_u + q_d = 1$. In particular we see that the weights $q_u$ and $q_d$ can be interpreted as probabilities for a new probability measure $Q$ with the property $Q(Z=u)=q_u$, $Q(Z=d)=q_d$. Denoting expectation w.r.t. this measure by $E^Q$ we now have the following easy calculation

$$\frac{1}{1+R}E^Q[S_1] = \frac{1}{1+R}[q_u su + q_d sd] = \frac{1}{1+R} \cdot s(1+R) = s.$$

We thus have the relation

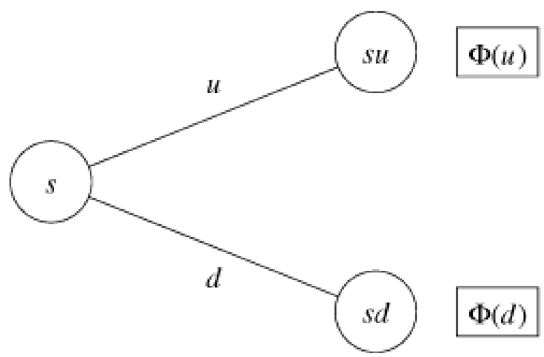
$$s = \frac{1}{1+R}E^Q[S_1],$$

which to an economist is a well known relation. It is in fact a **risk neutral** valuation formula, in the sense that it gives today's stock price as the discounted expected value of tomorrow's stock price. Of course we do not assume that the agents in our market are risk neutral — what we have shown is only that if we use the $Q$-probabilities instead of the objective probabilities then we have in fact a risk neutral valuation of the stock (given absence of arbitrage). A probability measure with this property is called a **risk neutral measure** , or alternatively a **risk adjusted measure** or a **martingale measure** . Martingale measures will play a dominant role in the sequel so we give a formal definition.

**Definition 2.4** *A probability measure $Q$ is called a* **martingale measure** *if the following condition holds.*

$$S_0 = \frac{1}{1+R}E^Q[S_1].$$

Fig. 2.2 Contingent Claim




We may now state the condition of no arbitrage in the following way.

**Proposition 2.5** *The market model is arbitrage free if and only if there exists a martingale measure Q.*

For the binomial model it is easy to calculate the martingale probabilities. The proof is left to the reader.

**Proposition 2.6** *For the binomial model above, the martingale probabilities are given by*

$$\begin{cases} q_u &= \dfrac{(1+R)-d}{u-d}, \\ q_d &= \dfrac{u-(1+R)}{u-d}. \end{cases}$$

## 2.1.3 Contingent Claims

Let us now assume that the market in the preceding section is arbitrage free. We go on to study pricing problems for contingent claims.

**Definition 2.7** *A* **contingent claim** *(financial derivative) is any stochastic variable $X$ of the form $X=\Phi(Z)$, where $Z$ is the stochastic variable driving the stock price process above.*

We interpret a given claim $X$ as a contract which pays $X$ SEK to the holder of the contract at time $t=1$. See Fig. 2.2, where the value of the claim at each node is given within the corresponding box. The function $\Phi$ is called the **contract function** . A typical example would be a European call option on the stock with strike price $K$. For this option to be interesting we assume that $sd<K<su$. If $S_1>K$ then we use the option, pay $K$ to get the stock and then sell the stock on the market for $su$, thus making a net profit of $su-K$. If $S_1<K$ then the option is obviously worthless. In this example we thus have

$$X = \begin{cases} su-K, & \text{if } Z = u, \\ 0, & \text{if } Z = d, \end{cases}$$

and the contract function is given by

$$\Phi(u) = su - K,$$
$$\Phi(d) = 0.$$

Our main problem is now to determine the "fair" price, if such an object exists at all, for a given contingent claim $X$. If we denote the price of $X$ at time $t$ by $\Pi\ (t; X)$, then it can be seen that at time $t = 1$ the problem is easy to solve. In order to avoid arbitrage we must (why?) have

$$\Pi\ (1; X) = X,$$

and the hard part of the problem is to determine $\Pi\ (0; X)$. To attack this problem we make a slight detour.

Since we have assumed absence of arbitrage we know that we cannot make money out of nothing, but it is interesting to study what we **can** achieve on the market.

**Definition 2.8** *A given contingent claim X is said to be* **reachable** *if there exists a portfolio h such that*

$$V_1^h = X,$$

*with probability 1. In that case we say that the portfolio h is a* **hedging** *portfolio or a* **replicating** *portfolio. If all claims can be replicated we say that the market is* **complete** .

If a certain claim $X$ is reachable with replicating portfolio $h$, then, from a financial point of view, there is no difference between holding the claim and holding the portfolio. No matter what happens on the stock market, the value of the claim at time $t$=1 will be exactly equal to the value of the portfolio at $t$=1. Thus the price of the claim should equal the market value of the portfolio, and we have the following basic pricing principle.

**Pricing principle 1** *If a claim X is reachable with replicating portfolio h, then the only reasonable price process for X is given by*

$$\Pi\ (t; X) = V_t^h, t = 0, 1.$$

The word "reasonable" above can be given a more precise meaning as in the following proposition. We leave the proof to the reader.

**Proposition 2.9** *Suppose that a claim X is reachable with replicating portfolio h. Then any price at t=0 of the claim X, other than* $V_0^h$ , *will lead to an arbitrage possibility.*

We see that in a complete market we can in fact price all contingent claims, so it is of great interest to investigate when a given market is complete. For the binomial model we have the following result.

**Proposition 2.10** *Assume that the general binomial model is free of arbitrage. Then it is also complete.*

**Proof** We fix an arbitrary claim $X$ with contract function $\Phi$, and we want to show that there exists a portfolio $h=(x,y)$ such that

$$\begin{aligned} V_1^h &= \Phi(u), \quad \text{if} \quad Z=u, \\ V_1^h &= \Phi(d), \quad \text{if} \quad Z=d. \end{aligned}$$

If we write this out in detail we want to find a solution $(x,y)$ to the following system of equations

$$\begin{aligned} (1+R)x + suy &= \Phi(u), \\ (1+R)x + sdy &= \Phi(d). \end{aligned}$$

Since by assumption $u>d$, this linear system has a unique solution, and a simple calculation shows that it is given by

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \tag{2.2}$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \tag{2.3}$$

## 2.1.4 Risk Neutral Valuation

Since the binomial model is shown to be complete we can now price any contingent claim. According to the pricing principle of the preceding section the price at $t=0$ is given by

$$\Pi\ (0; X) = V_0^h.$$

and using the explicit formulas (2.2)–(2.3) we obtain, after some reshuffling of terms,

$$\begin{aligned} \Pi\ (0, X) &= x + sy \\ &= \frac{1}{1+R}\left\{ \frac{(1+R)-d}{u-d} \cdot \Phi(u) + \frac{u-(1+R)}{u-d} \cdot \Phi(d) \right\}. \end{aligned}$$

Here we recognize the martingale probabilities $q_u$ and $q_d$ of Proposition 2.6. If we assume that the model is free of arbitrage, these are true probabilities (i.e. they are nonnegative), so we can write the pricing formula above as

$$\Pi\ (0; X) = \frac{1}{1+R} \{\Phi(u) \cdot q_u + \Phi(d) \cdot q_d\}.$$

The right hand side can now be interpreted as an expected value under the martingale probability measure $Q$, so we have proved the following basic pricing result, where we also add our old results about hedging.

**Proposition 2.11** *If the binomial model is free of arbitrage, then the arbitrage free price of a contingent claim X is given by*

$$\Pi\ (0; X) = \frac{1}{1+R} E^{Q}[X]. \tag{2.4}$$

*Here the martingale measure Q is uniquely determined by the relation*

$$S_0 = \frac{1}{1+R} E^{Q}[S_1], \tag{2.5}$$

*and the explicit expression for $q_u$ and $q_d$ are given in Proposition2.6. Furthermore the claim can be replicated using the portfolio*

$$x = \frac{1}{1+R} \cdot \frac{u\Phi(d) - d\Phi(u)}{u-d}, \tag{2.6}$$

$$y = \frac{1}{s} \cdot \frac{\Phi(u) - \Phi(d)}{u-d}. \tag{2.7}$$

We see that the formula (2.4) is a "risk neutral" valuation formula, and that the probabilities which are used are just those for which the stock itself admits a risk neutral valuation. The main economic moral can now be summarized.

**Moral:**

- When we compute the arbitrage free price of a financial derivative we carry out the computations **as if** we live in a risk neutral world.
- This does **not** mean that we *de facto* live (or believe that we live) in a risk neutral world.
- The valuation formula holds for all investors, regardless of their attitude towards risk, as long as they prefer more deterministic money to less.
- The formula above is therefore often referred to as a "preference free" valuation formula.

We end by studying a concrete example.

**Example 2.12** We set $s = 100$, $u = 1.2$, $d = 0.8$, $p_u$=0.6, $p_d$=0.4 and, for computational simplicity, R = 0. By convention, the monetary unit is the US dollar. Thus we have the price dynamics

$$S_0 = 100,$$
$$S_1 = \begin{cases} 120, \text{ with probability } 0.6, \\ 80, \text{ with probability } 0.4. \end{cases}$$

If we compute the discounted expected value (under the objective probability measure *P*) of tomorrow's price we get

$$\frac{1}{1+R} E^{P}[S_1] = 1 \cdot [120 \cdot 0.6 + 80 \cdot 0.4] = 104.$$

This is higher than the value of today's stock price of 100, so the market is risk averse. Since condition (2.1) obviously is satisfied we know that the market is arbitrage free. We consider a European call with strike price $K = 110$, so the claim $X$ is given by

$$X = \begin{cases} 10, & \text{if } S_1 = 120. \\ 0, & \text{if } S_1 = 80. \end{cases}$$

Using the method of computing the price as the discounted expected values under the objective probabilities, i.e. "Answer 1" in Section 1.1, this would give the price as

$$\Pi\ (0; X) = \frac{1}{1+0}[10 \cdot 0.6 + 0 \cdot 0.4] = 6.$$

Using the theory above it is easily seen that the martingale probabilities are given by $q_u = q_d = 0.5$, thus giving us the theoretical price

$$\Pi\ (0; X) = \frac{1}{1+0}[10 \cdot 0.5 + 0 \cdot 0.5] = 5.$$

We thus see that the theoretical price differs from the naive approach above. If our theory is correct we should also be able to replicate the option, and from the proposition above the replicating portfolio is given by

$$\begin{aligned} x &= \frac{1.2 \cdot 0 - 0.8 \cdot 10}{1.2 - 0.8} = -20, \\ y &= \frac{1}{100} \cdot \frac{10 - 0}{1.2 - 0.8} = \frac{1}{4}. \end{aligned}$$

In everyday terms this means that the replicating portfolio is formed by borrowing \$20 from the bank, and investing this money in a quarter of a share in the stock. Thus the net value of the portfolio at $t$=0 is five dollars, and at $t$=1 the value is given by

$$\begin{aligned} V_1^h &= -20 + \frac{1}{4} \cdot 120 = 10, \quad \text{if } S_1 = 120, \\ V_1^h &= -20 + \frac{1}{4} \cdot 80 = 0, \quad \text{if } S_1 = 80, \end{aligned}$$

so we see that we have indeed replicated the option. We also see that if anyone is foolish enough to buy the option from us for the price \$6, then we can make a riskless profit. We sell the option, thereby obtaining six dollars. Out of these six we invest five in the replicating portfolio and invest the remaining one in the bank. At time $t = 1$ the claims of the buyer of the option are completely balanced by the value of the replicating portfolio, and we still have one dollar invested in the bank. We have thus made an arbitrage profit. If someone is willing to sell the option to us at a price lower than five dollars, we can also make an arbitrage profit by selling the portfolio short.

We end this section by making some remarks.

First of all we have seen that in a complete market, like the binomial model above, there is indeed a unique price for any contingent claim. The price is given by the value of the replicating portfolio, and a negative way of expressing this is as follows. There exists a theoretical price for the claim precisely because of the fact that, strictly speaking, the claim is superfluous — it can equally well be replaced by its hedging portfolio.

Secondly we see that the structural reason for the completeness of the binomial model is the fact that we have two financial instruments at our disposal (the bond and the stock) in order to solve two equations (one for each possible outcome in the sample space). This fact can be generalized. A model is complete (in the generic case) if the number of underlying assets (including the bank account) equals the number of outcomes in the sample space.

If we would like to make a more realistic multiperiod model of the stock market, then the last remark above seems discouraging. If we make a (non-recombining) tree with 20 time steps this means that we have $2^{20} \sim 10^6$ elementary outcomes, and this number exceeds by a large margin the number of assets on any existing stock market. It would therefore seem that it is impossible to construct an interesting complete model with a reasonably large number of time steps. Fortunately the situation is not at all as bad as that; in a multiperiod model we will also have the possibility of considering **intermediary trading** , i.e. we can allow for portfolios which are rebalanced over time. This will give us much more degrees of freedom, and in the next section we will in fact study a complete multiperiod model.

# 2.2 The Multiperiod Model

## 2.2.1 Portfolios and Arbitrage

The multiperiod binomial model is a discrete time model with the time index $t$ running from $t = 0$ to $t = T$, where the horizon $T$ is fixed. As before we have two underlying assets, a bond with price process $B_t$ and a stock with price process $S_t$.

We assume a constant deterministic short rate of interest $R$, which is interpreted as the simple period rate. This means that the bond price dynamics are given by

$$\begin{aligned} B_{n+1} &= (1+R)B_n, \\ B_0 &= 1. \end{aligned}$$

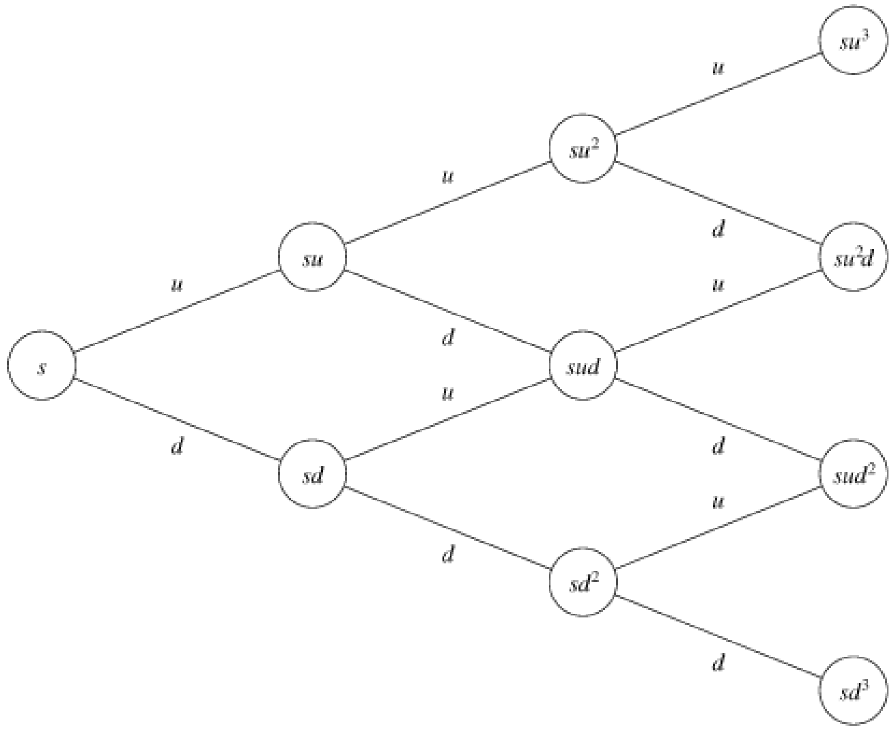
The dynamics of the stock price are given by

$$\begin{aligned} S_{n+1} &= S_n \cdot Z_n, \\ S_0 &= s. \end{aligned}$$

Here $Z_0, \ldots, Z_{T-1}$ are assumed to be i.i.d. (independent and identically distributed) stochastic variables, taking only the two values $u$ and $d$ with probabilities

$$P(Z_n = u) = p_u.$$

Fig. 2.3 Price Dynamics




$$P(Z_n = d) = p_d.$$

We can illustrate the stock dynamics by means of a tree, as in Fig. 2.3. Note that the tree is **recombining** in the sense that an "up"-move followed by a "down"-move gives the same result as a "down"-move followed by an "up"-move.

We now go on to define the concept of a dynamic portfolio strategy.

**Definition 2.13** *A* **portfolio strategy** *is a stochastic process*

$$\{h_t = (x_t, y_t);\ t = 1, \ldots, T\}$$

*such that $h_t$ is a function of $S_0, S_1, \ldots, S_{t-1}$. For a given portfolio strategy h we set $h_0 = h_1$ by convention. The* **value process** *corresponding to the portfolio h is defined by*

$$V_t^h = x_t(1 + R) + y_t S_t.$$

The interpretation of the formal definition is that $x_t$ is the amount of money which we invest in the bank at time $t-1$ and keep until time $t$. We interpret $y_t$ as the number of shares that we buy at time $t-1$ and keep until time $t$. We allow the portfolio strategy to be a contingent strategy, i.e. the portfolio we buy at $t$ is allowed to depend on all information we have collected by observing the evolution of the stock price up to time $t$. We are, however, not allowed to look into the future. The entity $V_t^h$ above is of course the market value of the portfolio $(x_t, y_t)$ (which has been held since $t-1$) at time $t$.

The portfolios which primarily interest us are the **self-financing** portfolios, i.e. portfolios without any exogenous infusion or withdrawal of money. In practical terms this means that in a self-financing portfolio strategy the accession of a new asset has to be financed through the sale of some other asset. The mathematical definition is as follows.

**Definition 2.14** *A portfolio strategy h is said to be* **self-financing** *if the following condition holds for all* $t=0,\ldots,T-1$

$$x_t(1+R)+y_tS_t=x_{t+1}+y_{t+1}S_t.$$

The condition above is simply a budget equation. It says that, at each time time $t$, the market value of the "old" portfolio $(x_t, y_t)$ (which was created at $t-1$) equals the purchase value of the new portfolio $(x_{t+1}, y_{t+1})$, which is formed at $t$ (and held until $t+1$).

We can now define the multiperiod version of an arbitrage possibility.

**Definition 2.15** *An* **arbitrage** *possibility is a self-financing portfolio h with the properties*

$$\begin{aligned} V_0^h &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

We immediately have the following necessary condition for absence of arbitrage.

**Lemma 2.16** *If the model is free of arbitrage then the following conditions necessarily must hold.*

$$d \leq (1+R) \leq u. \tag{2.8}$$

The condition above is in fact also sufficient for absence of arbitrage, but this fact is a little harder to show, and we will prove it later. In any case we assume that the condition holds.

**Assumption 2.2.1** *Henceforth we assume that $d < u$, and that the condition (2.8) holds.*

As in the one period model we will have use for "martingale probabilities" which are defined and computed exactly as before.

**Definition 2.17** *The martingale probabilities $q_u$ and $q_d$ are defined as the probabilities for which the relation*

$$s=\frac{1}{1+R}E^Q[\ S_{t+1}|S_t=s]$$

*holds.*

**Proposition 2.18** *The martingale probabilities are given by*

$$\begin{cases} q_u &= \dfrac{(1+R)-d}{u-d}, \\ q_d &= \dfrac{u-(1+R)}{u-d}. \end{cases}$$

## 2.2.2 Contingent Claims

We now give the formal definition of a contingent claim in the model.

**Definition 2.19** *A* **contingent claim** *is a stochastic variable $X$ of the form*

$$X = \Phi(S_T),$$

*where the* **contract function** $\Phi$ *is some given real valued function.*

The interpretation is that the holder of the contract receives the stochastic amount $X$ at time $t = T$. Notice that we are only considering claims that are "simple", in the sense that the value of the claim only depends on the value $S_T$ of the stock price at the final time $T$. It is also possible to consider stochastic payoffs which depend on the entire path of the price process during the interval $[0,T]$, but then the theory becomes a little more complicated, and in particular the event tree will become nonrecombining.

Our main problem is that of finding a "reasonable" price process

$$\{\Pi\ (t; X);\ t = 0, \ldots, T\}$$

for a given claim $X$, and as in the one period case we attack this problem by means of replicating portfolios.

**Definition 2.20** *A given contingent claim $X$ is said to be* **reachable** *if there exists a self-financing portfolio $h$ such that*

$$V_T^h = X,$$

*with probability 1. In that case we say that the portfolio $h$ is a* **hedging** *portfolio or a* **replicating** *portfolio. If all claims can be replicated we say that the market is (dynamically)* **complete** .

Again we have a natural pricing principle for reachable claims.

**Pricing principle 2** *If a claim $X$ is reachable with replicating (self-financing) portfolio $h$, then the only reasonable price process for $X$ is given by*

$$\Pi\ (t; X) = V_t^h,\ t = 0, 1, \ldots, T.$$

Let us go through the argument in some detail. Suppose that $X$ is reachable using the self-financing portfolio $h$. Fix $t$ and suppose that at time $t$ we have access to the amount $V_t^h$. Then we can invest this money in the portfolio $h$, and since the portfolio is self-financing we can rebalance it over time without any extra cost so as to have the stochastic value $V_T^h$ at time *T*. By definition $V_T^h = X$ with probability 1, so regardless of the stochastic movements of the stock price process the value of our portfolio will, at time *T*, be equal to the value of the claim *X*. Thus, from a financial point of view, the portfolio *h* and the claim *X* are equivalent so they should fetch the same price.

The "reasonableness" of the pricing formula above can be expressed more formally as follows. The proof is left to the reader.

**Proposition 2.21** *Suppose that X is reachable using the portfolio h. Suppose furthermore that, at some time t, it is possible to buy X at a price cheaper than (or to sell it at a price higher than)* $V_t^h$ *. Then it is possible to make an arbitrage profit.*

We now turn to the completeness of the model.

**Proposition 2.22** *The multiperiod binomial model is complete, i.e. every claim can be replicated by a self-financing portfolio.*

It is possible, and not very hard, to give a formal proof of the proposition, using mathematical induction. The formal proof will, however, look rather messy with lots of indices, so instead we prove the proposition for a concrete example, using a binomial tree. This should (hopefully) convey the idea of the proof, and the mathematically inclined reader is then invited to formalize the argument.

**Example 2.23** We set $T = 3$, $S_0=80$, $u = 1.5$, $d = 0.5$, $p_u=0.6$, $p_d=0.4$ and, for computational simplicity, $R = 0$.

The dynamics of the stock price can now be illustrated using the binomial tree in Fig. 2.4, where in each node we have written the value of the stock price.

We now consider a particular contingent claim, namely a European call on the underlying stock. The date of expiration of the option is $T = 3$, and the strike price is chosen to be $K = 80$. Formally this claim can be described as

$$X = \max[S_T - K, 0].$$

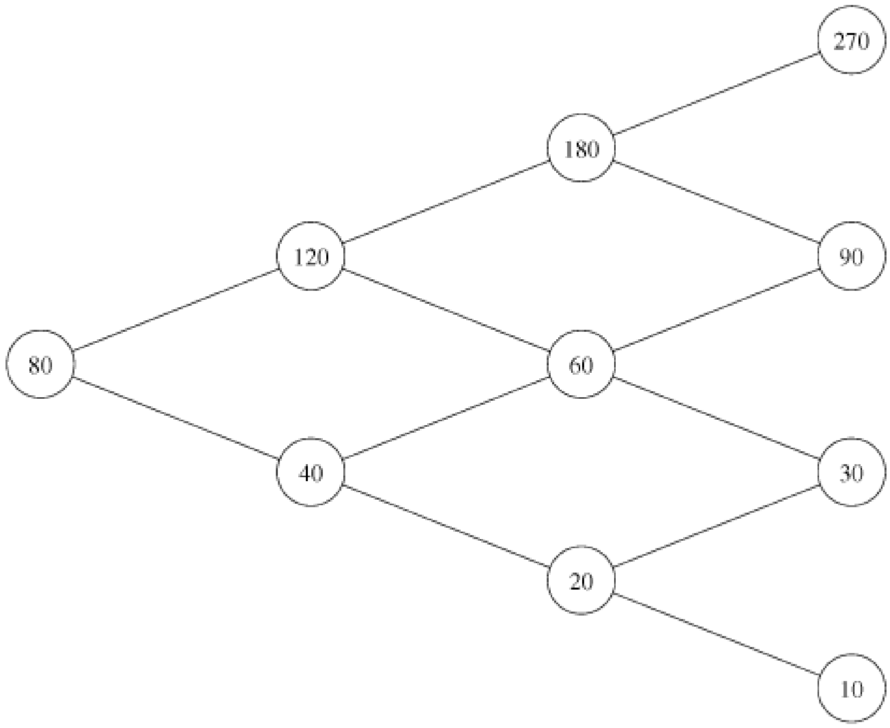
We will now show that this particular claim can be replicated, and it will be obvious from the argument that the result can be generalized to any binomial model and any claim.

The idea is to use induction on the time variable and to work backwards in the tree from the leaves at $t = T$ to the root at $t=0$. We start by computing the price of the option at the date of expiration. This is easily done since obviously (why?) we must have, for any claim *X*, the relation

$$\Pi\ (T; X) = X.$$

This result is illustrated in Fig. 2.5, where the boxed numbers indicate the price of the claim. Just to check, we see that if $S_3 = 90$, then we exercise the option,

Fig. 2.4 Price dynamics




pay 80 to obtain the stock, and then immediately sell the stock at market price 90, thus making a profit of 10.

Our problem is thus that of replicating the boxed payoff structure at $t = 3$. Imagine for a moment that we are at some node at $t = 2$, e.g. at the node $S_2 = 180$. What we then see in front of us, from this particular node, is a simple one period binomial model, given in Fig. 2.6, and it now follows directly from the one period theory that the payoff structure in Fig. 2.6 can indeed be replicated from the node $S_2 = 180$. We can in fact compute the cost of this replicating portfolio by risk neutral valuation, and since the martingale probabilities for this example are given by $q_u = q_d = 0.5$ the cost of the replicating portfolio is

$$\frac{1}{1+0}[190 \cdot 0.5 + 10 \cdot 0.5] = 100.$$

In the same way we can consider all the other nodes at $t = 2$, and compute the cost of the corresponding replicating portfolios. The result is the set of boxed numbers at $t = 2$ in Fig. 2.7.

What we have done by this procedure is to show that if we can find a self-financing portfolio which replicates the boxed payoff structure at $t = 2$, then it is in fact possible to replicate the original claim at $t = 3$. We have thus reduced the problem in the time variable, and from now on we simply reproduce the construction above, but this time at $t$=1. Take, for example, the node $S_1 = 40$. From the point of view of this node we have a one period model given by Fig.2.8,

Fig. 2.5

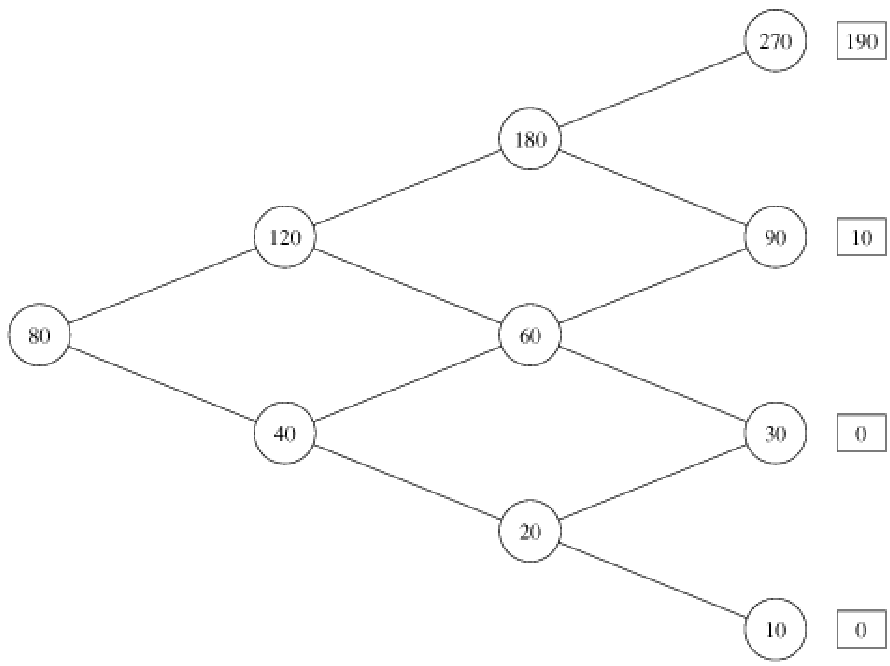
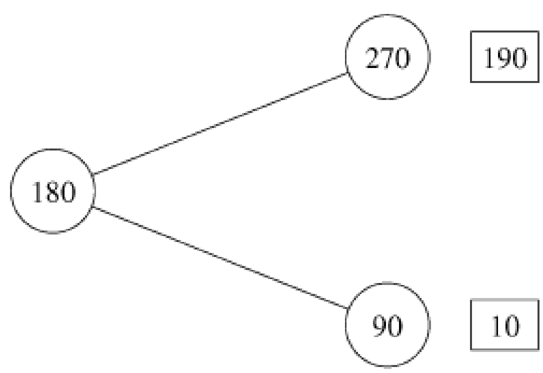



Fig. 2.6




and by risk neutral valuation we can replicate the payoff structure using a portfolio, which at the node $S_1 = 40$ will cost

$$\frac{1}{1+0}[5 \cdot 0.5 + 0 \cdot 0.5] = 2.5.$$

In this manner we fill the nodes at $t$=1 with boxed portfolio costs, and then we carry out the same construction again at $t$=0. The result is given in Fig. 2.9.

We have thus proved that it is in fact possible to replicate the European call option at an initial cost of 27.5. To check this let us now follow a possible price path forward through the tree.

Fig. 2.7

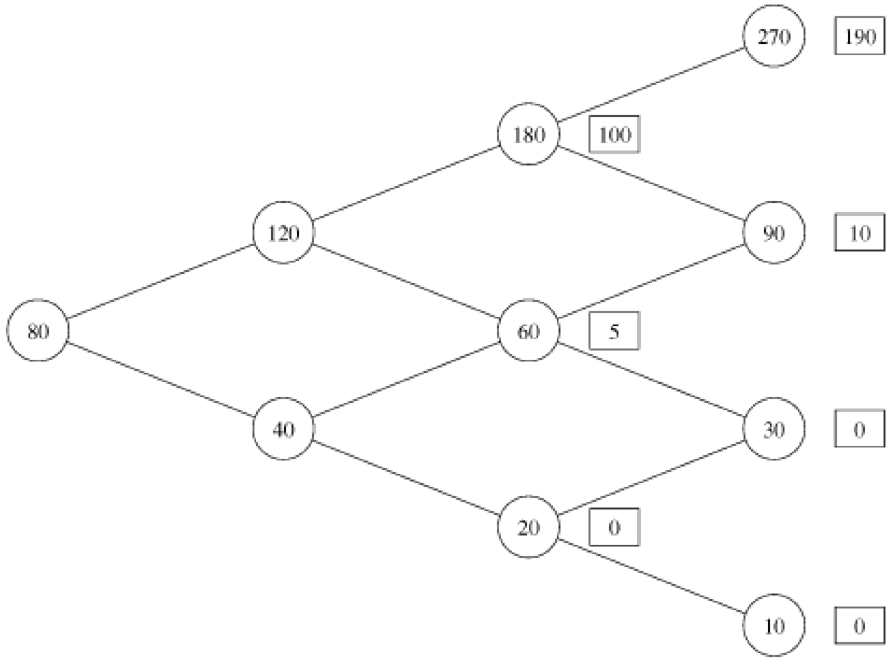
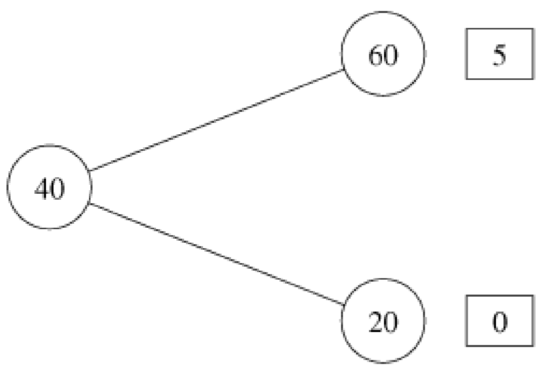



Fig. 2.8



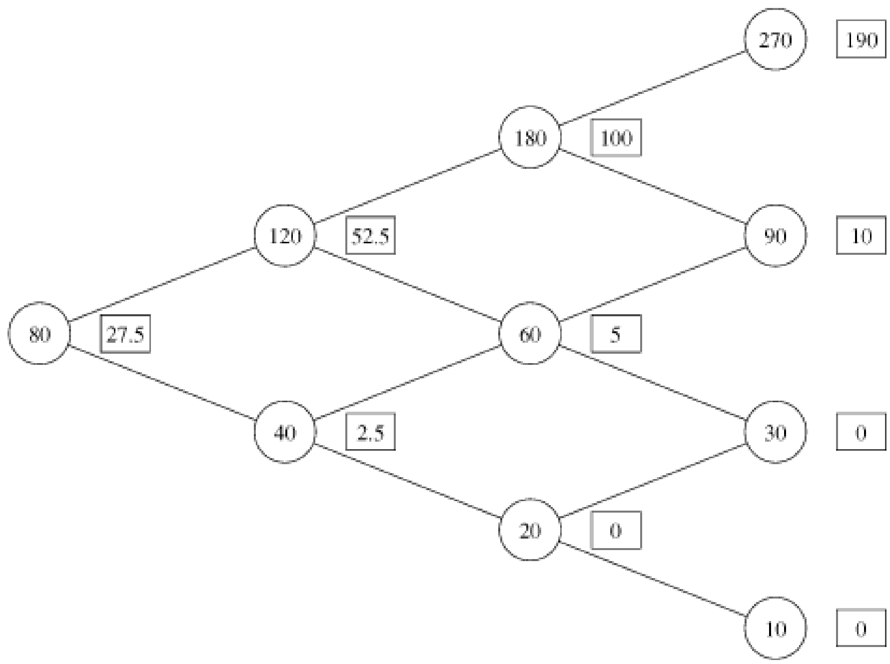

We start at $t$=0, and since we want to reproduce the boxed claim (52.5,2.5) at $t$=1, we can use Proposition 2.11 to compute the hedging portfolio as $x_1 = -22.5, y_1 = 5/8$. The reader should check that the cost of this portfolio is exactly 27.5.

Suppose that the price now moves to $S_1$=120. Then our portfolio is worth

$$-22.5 \cdot (1+0) + \frac{5}{8} \cdot 120 = 52.5.$$

Since we now are facing the claim (100,5) at $t = 2$ we can again use Proposition 2.4 to calculate the hedging portfolio as $x_2 = -42.5, y_2 = 95/120$, and the reader should again check that the cost of this portfolio equals the value of our

Fig. 2.9




old portfolio, i.e. 52.5. Thus it is really possible to rebalance the portfolio in a self-financing manner.

We now assume that the price falls to $S_2 = 60$. Then our portfolio is worth

$$-42.5 \cdot (1+0) + \frac{95}{120} \cdot 60 = 5.$$

Facing the claim (10,0) at $t = 3$ we use Proposition 2.4 to calculate the hedging portfolio as $x_3 = -5, y_3 = 1/6$, and again the cost of this portfolio equals the value of our old portfolio.

Now the price rises to $S_3 = 90$, and we see that the value of our portfolio is given by

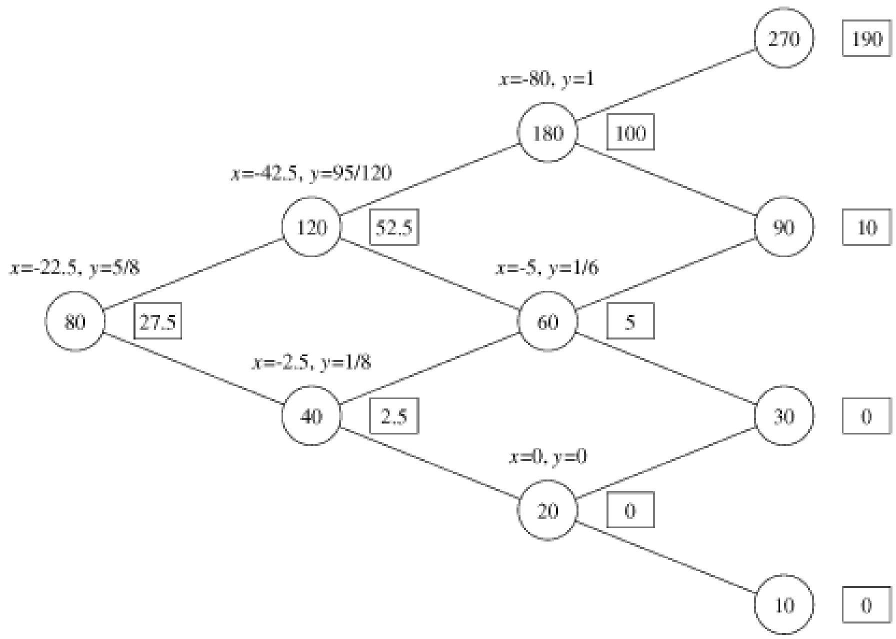
$$-5 \cdot (1+0) + \frac{1}{6} \cdot 90 = 10,$$

which is exactly equal to the value of the option at that node in the tree. In Fig. 2.10 we have computed the hedging portfolio at each node.

If we think about the computational effort we see that all the value computations, i.e. all the boxed values, have to be calculated off-line. Having done this we have of course not only computed the arbitrage free price at $t$=0 for the claim, but also computed the arbitrage free price, at every node in the tree.

The dynamic replicating portfolio does not have to be computed off-line. As in the example above, it can be computed on-line as the price process evolves over time. In this way we only have to compute the portfolio for those nodes that we actually visit.

Fig. 2.10




We now go on to give the general binomial algorithm. In order to do this we need to introduce some more notation to help us keep track of the price evolution. It is clear from the construction that the value of the price process at time $t$ can be written as

$$S_t = su^k d^{t-k}, k = 0, \ldots, t$$

where $k$ denotes the number of up-moves that have occurred. Thus each node in the binomial tree can be represented by a pair $(t,k)$ with $k = 0, \ldots, t$.

**Proposition 2.24 (Binomial algorithm)** *Consider a T-claim $X = \Phi(S_T)$. This claim can be replicated using a self-financing portfolio. If $V_t(k)$ denotes the value of the portfolio at the node $(t,k)$ then $V_t(k)$ can be computed recursively by the scheme*

$$\begin{cases} V_t(k) &= \frac{1}{1+R}\{q_u V_{t+1}(k+1) + q_d V_{t+1}(k)\}, \\ V_T(k) &= \Phi(su^k d^{T-k}), \end{cases}$$

*where the martingale probabilities $q_u$ and $q_d$ are given by*

$$\begin{cases} q_u &= \frac{(1+R)-d}{u-d}, \\ q_d &= \frac{u-(1+R)}{u-d}. \end{cases}$$

*With the notation as above, the hedging portfolio is given by*

$$\begin{cases} x_t(k) &= \frac{1}{1+R} \cdot \frac{uV_t(k) - dV_t(k+1)}{u-d}, \\ y_t(k) &= \frac{1}{S_{t-1}} \cdot \frac{V_t(k+1) - V_t(k)}{u-d} \end{cases}$$

*In particular, the arbitrage free price of the claim at* $t=0$ *is given by* $V_0(0)$.

From the algorithm above it is also clear that we can obtain a risk neutral valuation formula.

**Proposition 2.25** *The arbitrage free price at* $t=0$ *of a T-claim* $X$ *is given by*

$$\Pi\ (0; X) = \frac{1}{(1+R)^T} \cdot E^Q[X],$$

*where* $Q$ *denotes the martingale measure, or more explicitly*

**Proof** The first formula follows directly from the algorithm above. If we let $Y$ denote the number of up-moves in the tree we can write

$$X = \Phi(S_T) = \Phi(su^Y d^{T-Y}),$$

and now the second formula follows from the fact that $Y$ has a binomial distribution.

We end this section by proving absence of arbitrage.

**Proposition 2.26** *The condition*

$$d < (1+R) < u$$

*is a necessary and sufficient condition for absence of arbitrage.*

**Proof** The necessity follows from the corresponding one period result. Assume that the condition is satisfied. We want to prove absence of arbitrage, so let us assume that $h$ (a potential arbitrage portfolio) is a self-financing portfolio satisfying the conditions

$$\begin{aligned} P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

From these conditions, and from the risk neutral valuation formula, it follows that

$$V_0^h = \frac{1}{(1+R)^T} \cdot E^Q[V_T^h] > 0,$$

which shows that $h$ is not an arbitrage portfolio.

## 2.3 Exercises

**Exercise 2.1**

(a) Prove Proposition 2.6.
(b) Show, in the one period binomial model, that if $\Pi(1; X) \neq X$ with probability 1, then you can make a riskless profit.

**Exercise 2.2** Prove Proposition 2.21.

**Exercise 2.3** Consider the multiperiod example in the text. Suppose that at time $t=1$ the stock price has gone up to 120, and that the market price of the option turns out to be 50.0. Show explicitly how you can make an arbitrage profit.

**Exercise 2.4** Prove Proposition 2.24, by using induction on the time horizon $T$.

## 2.4 Notes

For the origins of the binomial model, see Cox and Rubenstein (1979), and Rendleman and Bartter (1979). The book by Cox and Rubenstein (1985) has become a standard reference.

# 3 Stochastic Integrals

## 3.1 Introduction

The purpose of this book is to study asset pricing on financial markets in continuous time. We thus want to model asset prices as continuous time stochastic processes, and the most complete and elegant theory is obtained if we use **diffusion processes** and **stochastic differential equations** as our building blocks. What, then, is a diffusion?

Loosely speaking we say that a stochastic process $X$ is a diffusion if its local dynamics can be approximated by a stochastic difference equation of the following type.

$$X(t+\Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))Z(t). \tag{3.1}$$

Here $Z(t)$ is a normally distributed disturbance term which is independent of everything which has happened up to time $t$, while $\mu$ and $\sigma$ are given deterministic functions. The intuitive content of (3.1) is that, over the time interval $[t, t + \Delta t]$, the $X$-process is driven by two separate terms.

- A locally deterministic velocity $\mu(t,X(t))$.
- A Gaussian disturbance term, amplified by the factor $\sigma(t,X(t))$.

The function $\mu$ is called the (local) **drift** term of the process, whereas $\sigma$ is called the **diffusion** term. In order to model the Gaussian disturbance terms we need the concept of a Wiener process.

**Definition 3.1** *A stochastic process $W$ is called a* **Wiener process** *if the following conditions hold.*

1. $W(0)=0$.
2. *The process $W$ has independent increments, i.e. if $r<s \le t < u$ then $W(u)-W(t)$ and $W(s)-W(r)$ are independent stochastic variables.*
3. *For $s < t$ the stochastic variable $W(t)-W(s)$ has the Gaussian distribution* $N[0,\sqrt{t-s}]$.
4. *$W$ has continuous trajectories.*

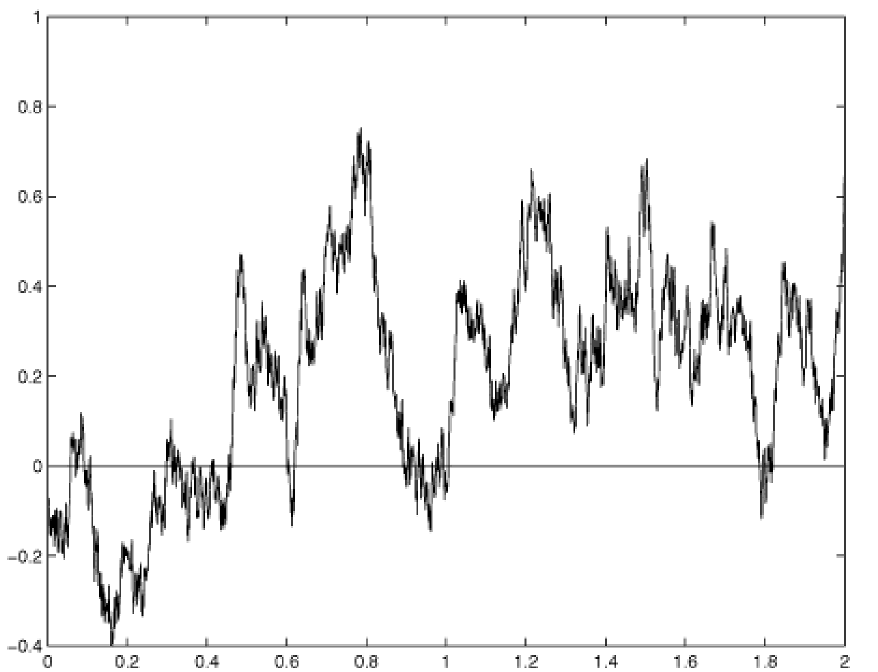
In Fig. 3.1 a computer simulated Wiener trajectory is shown.

We may now use a Wiener process in order to write (3.1) as

$$X(t+\Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))\Delta W(t), \tag{3.2}$$

where $\Delta W(t)$ is defined by

Fig. 3.1 A Wiener Trajectory




$$\Delta W(t) = W(t + \Delta t) - W(t).$$

Let us now try to make (3.2) a bit more precise. It is then tempting to divide the equation by $\Delta t$ and let $\Delta t$ tend to zero. Formally we would obtain

$$\dot{X}(t) = \mu(t, X(t)) + \sigma(t, X(t))v(t),$$

$$X(0) = a, \tag{3.3}$$

where we have added an initial condition and where (3.4)

$$v(t) = \frac{dW}{dt}$$

is the formal time derivative of the Wiener process $W$.

If $v$ were an ordinary (and well defined) process we would now in principle be able to solve (3.3) as a standard ordinary differential equation (ODE) for each $v$-trajectory. However, it can be shown that with probability 1 a Wiener trajectory is nowhere differentiable (cf. Fig. 3.1), so the process $v$ cannot even be defined. Thus this is a dead end.

Another possibility of making eqn (3.2) more precise is to let $\Delta t$ tend to zero without first dividing the equation by $\Delta t$. Formally we will then obtain the expression

$$\begin{cases} dX(t) & = \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X(0) & = a, \end{cases} \tag{3.5}$$

and it is now natural to interpret (3.5) as a shorthand version of the following integral equation

$$X(t) = a + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s). \tag{3.6}$$

In eqn (3.6) we may interpret the *ds*-integral as an ordinary Riemann integral. The natural interpretation of the *dW*-integral is to view it as a Riemann–Stieltjes integral for each *W*-trajectory, but unfortunately this is not possible since one can show that the *W*-trajectories are of locally unbounded variation. Thus the stochastic *dW*-integral cannot be defined in a naive way.

As long as we insist on giving a precise meaning to eqn (3.2) **for each *W*-trajectory separately**, we thus seem to be in a hopeless situation. If, however, we relax our demand that the *dW*-integral in eqn (3.6) should be defined trajectorywise we can still proceed. It is in fact possible to give a global ($L^2$-)definition of integrals of the form

$$\int_0^t g(s)dW(s) \tag{3.7}$$

for a large class of processes *g*. This new integral concept—the so called Itô integral—will then give rise to a very powerful type of stochastic differential calculus—the Itô calculus. Our program for the future thus consists of the following steps.

1. Define integrals of the type

$$\int_0^t g(s)dW(s).$$

2. Develop the corresponding differential calculus.
3. Analyze stochastic differential equations of the type (3.5) using the stochastic calculus above.

## 3.2 Information

Let $X$ be any given stochastic process. In the sequel it will be important to define "the information generated by $X$" as time goes by. To do this in a rigorous fashion is unfortunately outside the scope of this book, but for most practical purposes the following heuristic definitions will do nicely.

**Definition 3.2** *The symbol $\mathcal{F}_t^X$ denotes "the information generated by $X$ on the interval $[0,t]$", or alternatively "what has happened to $X$ over the interval $[0,t]$". If, based upon observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, it is possible to decide whether a given event $A$ has occurred or not, then we write this as*

$$A \in \mathcal{F}_t^X.$$

*If the value of a given stochastic variable $Z$ can be completely determined given observations of the trajectory $\{X(s); 0 \leq s \leq t\}$, then we also write*

$$Z \in \mathcal{F}_t^X.$$

*If Y is a stochastic process such that we have*

$$Y(t) \in \mathcal{F}_t^X$$

*for all* $t \geq 0$ *then we say that* $Y$ *is* **adapted** *to the* **filtration** $(\mathcal{F}_t^X)_{t \geq 0}$. The above definition is only intended to have an intuitive content, since a precise definition would take us into the realm of abstract measure theory. Nevertheless it is usually extremely simple to use the definition, and we now give some fairly typical examples.

1. If we define the event $A$ by $A = \{X(s) \leq 3.14, \text{ for all } s \leq 9\}$ then we have $A \in \mathcal{F}_9^X$.
2. For the event $A = \{X(10) > 8\}$ we have $A \in \mathcal{F}_{10}^X$. Note, however, that we do not have $A \in \mathcal{F}_9^X$, since it is impossible to decide if $A$ has occurred or not on the basis of having observed the $X$-trajectory only over the interval [0,9].
3. For the stochastic variable $Z$, defined by

$$Z = \int_0^5 X(s)ds,$$

we have $Z \in \mathcal{F}_5^X$.

4. If $W$ is a Wiener process and if the process $X$ is defined by

$$X(t) = \sup_{s \leq t} W(s)$$

then $X$ is adapted to the $W$-filtration.

5. With $W$ as above, but with $X$ defined as

$$X(t) = \sup_{s \leq t+1} W(s),$$

$X$ is not adapted (to the $W$-filtration).

# 3.3 Stochastic Integrals

We now turn to the construction of the stochastic integral. For that purpose we consider as given a Wiener process $W$, and another stochastic process $g$. In order to guarantee the existence of the stochastic integral we have to impose some kind of integrability conditions on $g$, and the class $\mathcal{L}^2$ turns out to be natural.

**Definition 3.3**

*(i) We say that the process g belongs to the class* $\mathcal{L}^2$ $[a,b]$ *if the following conditions are satisfied.*

- $\int_a^b E[g^2(s)]ds < \infty$.

- *The process g is adapted to the $\mathcal{F}_t^W$-filtration.*

*(ii) We say that the process g belongs to the class $\mathcal{L}^2$ if $g \in \mathcal{L}^2[0,t]$ for all $t>0$.*

Our object is now to define the stochastic integral $\int_a^b g(s)dW(s)$, for a process $g \in \mathcal{L}^2$ $[a,b]$, and this is carried out in two steps.

Suppose to begin with that the process $g \in \mathcal{L}^2$ $[a,b]$ is **simple** , i.e. that there exist deterministic points in time $a = t_0 < t_1 < \cdots < t_n = b$, such that $g$ is constant on each subinterval. In other words we assume that $g(s) = g(t_k)$ for $s \in [t_k, t_{k+1})$. Then we define the stochastic integral by the obvious formula

$$\int_a^b g(s)dW(s) = \sum_{k=0}^{n-1} g(t_k)\,[W(t_{k+1}) - W(t_k)]. \tag{3.8}$$

**Remark 3.3.1** Note that in the definition of the stochastic integral we take so called **forward** increments of the Wiener process. More specifically, in the generic term $g(t_k)[W(t_{k+1}) - W(t_k)]$ of the sum the process $g$ is evaluated at the **left** end $t_k$ of the interval $[t_k, t_{k+1}]$ over which we take the $W$-increment. This is essential to the following theory both from a mathematical and (as we shall see later) from an economical point of view.

For a general process $g \in \mathcal{L}^2[a,b]$ which is not simple we may schematically proceed as follows.

1. Approximate $g$ with a sequence of simple processes $g_n$ such that

$$\int_a^b E\left[\{g_n(s) - g(s)\}^2\right]ds \to 0.$$

2. For each $n$ the integral $\int_a^b g_n(s)dW(s)$ is a well defined stochastic variable $Z_n$, and it is possible to prove that there exists a stochastic variable $Z$ such that $Z_n \to Z$ (in $L^2$) as $n \to \infty$.
3. We now define the stochastic integral by

$$\int_a^b g(s)dW(s) = \lim_{n\to\infty}\int_a^b g_n(s)dW(s). \tag{3.9}$$

The most important properties of the stochastic integral are given by the following proposition. In particular we will use the property (3.12) over and over again.

**Proposition 3.4** *Let g be a process satisfying the conditions*

$$\int_a^b E[g^2(s)]ds < \infty. \tag{3.10}$$

$$g \text{ is adapted to the } \mathcal{F}_t^W\text{-filtration.} \tag{3.11}$$

*Then the following relations hold*

$$E\left[\int_a^b g(s)dW(s)\right] = 0. \tag{3.12}$$

$$E\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = \int_a^b E[g^2(s)]ds \tag{3.13}$$

$$\int_a^b g(s)dW(s) \quad is \quad \mathcal{F}_b^W\text{-measurable}. \tag{3.14}$$

**Proof** A full proof is outside the scope of this book, but the general strategy is to start by proving all the assertions above in the case when $g$ is simple. This is fairly easily done, and then it "only" remains to go to the limit in the sense of (3.9). We illustrate the technique by proving (3.12) in the case of a simple $g$. We obtain

$$\begin{aligned} E\left[\int_a^b g(s)dW(s)\right] &= E\left[\sum_{k=0}^{n-1} g(t_k)[W(t_{k+1}) - W(t_k)]\right] \\ &= \sum_{k=0}^{n-1} E[g(t_k)[W(t_{k+1}) - W(t_k)]]. \end{aligned}$$

Since $g$ is adapted, the value $g(t_k)$ only depends on the behavior of the Wiener process on the interval $[0,t_k]$. Now, by definition $W$ has independent increments, so $[W(t_{k+1}) - W(t_k)]$ (which is a **forward** increment) is independent of $g(t_k)$. Thus we have

$$\begin{aligned} E[g(t_k)[W(t_{k+1}) - W(t_k)]] &= E[g(t_k)] \cdot E[W(t_{k+1}) - W(t_k)] \\ &= E[g(t_k)] \cdot 0 = 0. \end{aligned}$$

**Remark 3.3.2** It is possible to define the stochastic integral for a process $g$ satisfying only the weak condition

$$P\left(\int_a^b g^2(s)ds < \infty\right) = 1. \tag{3.15}$$

For such a general $g$ we have no guarantee that the properties (3.12) and (3.13) hold. Property (3.14) is, however, still valid.

## 3.4 Martingales

The theory of stochastic integration is intimately connected to the theory of martingales, and the modern theory of financial derivatives is in fact based mainly on martingale theory. Unfortunately for us, martingale theory requires some basic knowledge of abstract measure theory, and a formal treatment is thus outside the scope of this book.

Because of its great importance for the field, however, it would be unreasonable to pass over this important topic entirely, and the object of this section is to (informally) introduce the martingale concept.

Let us therefore consider a given filtration ("flow of information") $(\mathcal{F}_t)_{t\geq 0}$, where, as before, the reader can think of $\mathcal{F}_t$ as the information generated by all observed events up to time *t*. For any stochastic variable *Y* we now let the symbol

$$E[\,Y|\mathcal{F}_t]$$

denote the "expected value of *Y*, given the information available at time *t*". A precise definition of this object requires measure theory, so we have to be content with this informal description. Note that for a fixed *t*, the object $E[\,Y|\mathcal{F}_t]$ is a stochastic variable. If, for example, the filtration is generated by a single observed process *X*, then the information available at time *t* will of course depend upon the behaviour of *X* over the interval [0, *t*], so the conditional expectation $E[\,Y|\mathcal{F}_t]$ will in this case be a function of all past *X*-values $\{X(s): s \leq t\}$. We will need the following two rules of calculation.

**Proposition 3.5**

- *If Y and Z are stochastic variables, and Z is $\mathcal{F}_t$-measurable, then*

$$E[\,Z\cdot Y|\mathcal{F}_t] = Z\cdot E[\,Y|\mathcal{F}_t].$$

- *If Y is a stochastic variable, and if $s < t$, then*

$$E[E\,[\,Y|\mathcal{F}_t]|\mathcal{F}_s] = E[\,Y|\mathcal{F}_s].$$

The first of these results should be obvious: in the expected value $E[\,Z\cdot Y|\mathcal{F}_t]$ we condition upon all information available at time *t*. If now $Z\in\mathcal{F}_t$, this means that, given the information $\mathcal{F}_t$, we know exactly the value of *Z*, so in the conditional expectation *Z* can be treated as a constant, and thus it can be taken outside the expectation. The second result is called the "law of iterated expectations", and it is basically a version of the law of total probability.

We can now define the martingale concept.

**Definition 3.6** *A stochastic process X is called an* $(\mathcal{F}_t)$-**martingale** *if the following conditions hold.*

- *X is adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$.
- *For all t*

$$E[|X(t)|] < \infty.$$

- *For all s and t with $s \leq t$ the following relation holds*

$$E[X(t)|\mathcal{F}_s] = X(s).$$

*A process satisfying, for all s and t with $s \leq t$, the inequality*

$$E[X(t)|\mathcal{F}_s] \leq X(s),$$

*is called a* **supermartingale** , *and a process satisfying*

$$E[X(t)|\mathcal{F}_s] \geq X(s),$$

*is called a* **submartingale** .

The first condition says that we can observe the value $X(t)$ at time $t$, and the second condition is just a technical condition. The really important condition is the third one, which says that the expectation of a future value of $X$, given the information available today, equals today's observed value of $X$. Another way of putting this is to say that a martingale has no systematic drift.

It is possible to prove the following extension of Proposition 3.4.

**Proposition 3.7** *For any process $g \in \mathcal{L}^2[s,t]$ the following hold.*

$$E\left[\int_s^t g(u)dW(u)\middle|\mathcal{F}_s^W\right] = 0.$$

As a corollary we obtain the following important fact. corollary 3.8 *For any process $g \in \mathcal{L}^2$, the process X, defined by*

$$X(t) = \int_0^t g(s)dW(s),$$

*is an $(\mathcal{F}_t^W)$-martingale. In other words, modulo an integrability condition,* **every stochastic integral is a martingale** .

**Proof** Fix $s$ and $t$ with $s < t$. We have

$$\begin{aligned} E[X(t)|\mathcal{F}_s^W] &= E\left[\int_0^t g(\tau)dW(\tau)\middle|\mathcal{F}_s^W\right] \\ &= E\left[\int_0^s g(\tau)dW(\tau)\middle|\mathcal{F}_s^W\right] + E\left[\int_s^t g(\tau)dW(\tau)\middle|\mathcal{F}_s^W\right]. \end{aligned}$$

The integral in the first expectation is, by Proposition 3.4, measurable w.r.t. $\mathcal{F}_s^W$, so by Proposition 3.5 we have

$$E\left[\int_0^s g(\tau)dW(\tau)\middle|\mathcal{F}_s^W\right] = \int_0^s g(\tau)dW(\tau).$$

From Proposition 3.4 we also see that $E\left[\int_s^t g(\tau)dW(\tau)\middle|\mathcal{F}_s^W\right] = 0$, so we obtain

$$E[X(t)|\mathcal{F}_s^W] = \int_0^s g(\tau)dW(\tau) + 0 = X(s).$$

We have in fact the following stronger (and very useful) result.

**Lemma 3.8** *Within the framework above, and assuming enough integrability, a stochastic process X (having a stochastic differential) is a martingale if and only if the stochastic differential has the form*

$$dX(t) = g(t)dW(t),$$

*i.e. X has no dt-term.*

**Proof** We have already seen that if $dX$ has no $dt$-term then $X$ is a martingale. The reverse implication is much harder to prove, and the reader is referred to the literature cited in the notes below.

# 3.5 Stochastic Calculus and the Itô Formula

Let $X$ be a stochastic process and suppose that there exists a real number $x_0$ and two adapted processes μ and σ such that the following relation holds for all $t \geq 0$.

$$X(t) = x_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s). \tag{3.16}$$

As usual $W$ is a Wiener process. To use a less cumbersome notation we will often write eqn (3.16) in the following form

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \tag{3.17}$$

$$X(0) = a. \tag{3.18}$$

In this case we say that $X$ has a **stochastic differential** given by (3.17) with an **initial condition** given by (3.18). Note that the formal string $dX(t) = \mu(t)dt + \sigma(t)dW(t)$ has no independent meaning. It is simply a shorthand version of the expression (3.16) above. From an intuitive point of view the stochastic differential is, however, a much more natural object to consider than the corresponding integral expression. This is because the stochastic differential gives us the "infinitesimal dynamics" of the $X$-process, and as we have seen in Section 3.1 both the drift term $\mu(s)$ and the diffusion term $\sigma(s)$ have a natural intuitive interpretation.

Let us assume that $X$ indeed has the stochastic differential above. Loosely speaking we thus see that the infinitesimal increment $dX(t)$ consists of a locally deterministic drift term $\mu(t)dt$ plus an **additive Gaussian** noise term $\sigma(t)dW(t)$. Assume furthermore that we are given a $C^{1,2}$-function

$$f: R_+ \times R \to R$$

and let us now define a new process $Z$ by

$$Z(t) = f(t, X(t)).$$

We may now ask what the local dynamics of the $Z$-process look like, and at first it seems fairly obvious that, except for the case when $f$ is linear in $x$, $Z$ will not have a stochastic differential. Consider, for example, a discrete time example where $X$ satisfies a stochastic difference equation with additive Gaussian noise in each step, and suppose that $f(t,x)=e^x$. Then it is clear that $Z$ will not be driven by additive Gaussian noise—the noise will in fact be multiplicative and log-normal. It is therefore extremely surprising that for continuous time models the stochastic differential structure with a drift term plus additive Gaussian noise will in fact be preserved even under nonlinear transformations. Thus the process $Z$ **will** have a stochastic differential, and the form of $dZ$ is given explicitly by the famous Itô formula below. Before turning to the Itô formula we have to take a closer look at some rather fine properties of the trajectories of the Wiener process.

As we saw earlier the Wiener process is defined by a number of very simple probabilistic properties. It is therefore natural to assume that a typical Wiener trajectory is a fairly simple object, but this is not at all the case. On the contrary—one can show that, with probability 1, the Wiener trajectory will be a continuous function of time (see the definition above) which is nondifferentiable at every point. Thus a typical trajectory is a continuous curve consisting entirely of corners and it is of course quite impossible to draw a figure of such an object (it is in fact fairly hard to prove that such a curve actually exists). This lack of smoothness gives rise to an odd property of the quadratic variation of the Wiener trajectory, and since the entire theory to follow depends on this particular property we now take some time to study the Wiener increments a bit closer.

Let us therefore fix two points in time, $s$ and $t$ with $s < t$, and let us use the handy notation

$$\begin{aligned} \Delta t &= t - s, \\ \Delta W(t) &= W(t) - W(s). \end{aligned}$$

Using well known properties of the normal distribution it is fairly easy to obtain the following results, which we will use frequently.

$$E[\Delta W] = 0, \tag{3.19}$$

$$E\left[(\Delta W)^2\right] = \Delta t, \tag{3.20}$$

$$Var[\Delta W] = \Delta t, \tag{3.21}$$

$$Var[(\Delta W)^2] = 2(\Delta t)^2. \tag{3.22}$$

We see that the squared Wiener increment $(\Delta W(t))^2$ has an expected value which equals the time increment $\Delta t$. The really important fact, however, is that, according to (3.22), the variance of $[\Delta W(t)]^2$ is negligible compared to its expected value. In other words, as $\Delta t$ tends to zero $[\Delta W(t)]^2$ will of course also tend to zero, but the variance will approach zero much faster than the expected value. Thus $[\Delta W(t)]^2$ will look more and more "deterministic" and we are led to believe that in the limit we have the purely formal equality

$$[dW(t)]^2 = dt. \tag{3.23}$$

The reasoning above is purely heuristic. It requires a lot of hard work to turn the relation (3.23) into a mathematically precise statement, and it is of course even harder to prove it. We will not attempt either a precise formulation or a precise proof. In order to give the reader a flavor of the full theory we will, however, give another argument for the relation (3.23).

Let us therefore fix a point in time $t$ and subdivide the interval $[0,t]$ into $n$ equally large subintervals of the form $[k\frac{t}{n},(k+1)\frac{t}{n}]$, where $k = 0,1, \ldots, n-1$. Given this subdivision, we now define the quadratic variation of the Wiener process by $S_n$, i.e.

$$S_n = \sum_{i=1}^{n}\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2, \tag{3.24}$$

and our goal is to see what happens to $S_n$ as the subdivision becomes finer, i.e. as $n \to \infty$. We immediately see that

$$\begin{aligned} E[S_n] &= \sum_{i=1}^{n} E\left[\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right] \\ &= \sum_{i=1}^{n}\left[i\frac{t}{n} - (i-1)\frac{t}{n}\right] = t. \end{aligned}$$

Using the fact that $W$ has independent increments we also have

$$\begin{aligned} Var[S_n] &= \sum_{i=1}^{n} Var\left[\left[W\left(i\frac{t}{n}\right) - W\left((i-1)\frac{t}{n}\right)\right]^2\right] \\ &= \sum_{i=1}^{n} 2\left[\frac{t^2}{n^2}\right] = \frac{2t^2}{n}. \end{aligned}$$

Thus we see that $E[S_n] = t$ whereas $Var[S_n] \to 0$ as $n \to \infty$. In other words, as $n \to \infty$ we see that $S_n$ tends to the **deterministic** limit $t$. This motivates us to write

$$\int_0^t [dW]^2 = t, \tag{3.25}$$

or, equivalently,

$$[dW]^2 = dt. \tag{3.26}$$

Note again that all the reasoning above has been purely motivational. In this text we will have to be content with accepting (3.26) as a dogmatic truth, and now we can give the main result in the theory of stochastic calculus—the Itô formula.

**Theorem 3.10 (Itô's formula)** *Assume that the process X has a stochastic differential given by*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t). \tag{3.27}$$

*where* $\mu$ *and* $\sigma$ *are adapted processes, and let f be a* $C^{1,2}$*-function. Define the process Z by* $Z(t) = f(t, X(t))$*. Then Z has a stochastic differential given by*

$$df(t, X(t)) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t) \tag{3.28}$$

**Remark 3.5.1** In the statement of the theorem above we have, for readability reasons, suppressed a lot of variables. The term $\mu \frac{\partial f}{\partial x}$, for example, is shorthand notation for

$$\mu(t) \frac{\partial f}{\partial x}(t, X(t))$$

and correspondingly for the other terms.

**Proof** A full formal proof is outside the scope of this text, so we only give a heuristic proof. (See Remark 3.5.2 below.) If we make a Taylor expansion including second order terms we obtain

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 + \frac{\partial^2 f}{\partial t \partial x} dt dX. \tag{3.29}$$

By definition we have

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

so, at least formally, we obtain

$$(dX)^2 = \mu^2 (dt)^2 + 2\mu\sigma(dt)(dW) + \sigma^2 (dW)^2.$$

The term containing $(dt)^2$ above is negligible compared to the $dt$-term in (3.27), and it can also be shown that the $(dt)(dW)$-term is negligible compared to the $dt$-term. Furthermore we have $(dW)^2 = dt$ from (3.23), and plugging in all this into the Taylor expansion (3.29) gives us the result.

It may be hard to remember the Itô formula, so for practical purposes it is often easier to copy our "proof" above and make a second order Taylor expansion.

**Proposition 3.11 (Itô's formula)** *With assumptions as in Theorem 3.10, df is given by*

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 \tag{3.30}$$

*where we use the following formal multiplication table.*

$$\begin{cases} (dt)^2 & = 0, \\ dt \cdot dW & = 0, \\ (dW)^2 & = dt. \end{cases}$$

**Remark 3.5.2** As we have pointed out, the "proof" of the Itô formula above does not at all constitute a formal proof. We end this section by giving an outline of the full proof. What we have to prove is that, for all $t$, the following relation holds with probability one.

$$\begin{aligned} f(t,X(t)) - f(0,X(0)) &= \int_0^t \left( \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} \right) ds \\ &+ \int_0^t \sigma \frac{\partial f}{\partial x} dW(s). \end{aligned}$$

(3.31)

We therefore divide the interval $[0,t]$ as $0 = t_0 < t_1 < \ldots < t_n = t$ into $n$ equal subintervals. Then we have

$$f(t,X(t)) - f(0,X(0)) = \sum_{k=0}^{n-1} f(t_{k+1}, X(t_{k+1})) - f(t_k, X(t_k)).$$

Using Taylor's formula we obtain, with subscripts denoting partial derivatives and obvious notation, (3.32)

$$\begin{aligned} f(t_{k+1}, X(t_{k+1})) - f(t_k, X(t_k)) &= f_t(t_k, X(t_k))\Delta t + f_x(t_k, X(t_k))\Delta X_k \\ &+ \frac{1}{2} f_{xx}(t_k, X(t_k))(\Delta X_k)^2 + Q_k, \end{aligned}$$

where $Q_k$ is the remainder term. Furthermore we have (3.33)

$$\begin{aligned} \Delta X_k &= X(t_{k+1}) - X(t_k) = \int_{t_k}^{t_{k+1}} \mu(s) ds + \int_{t_k}^{t_{k+1}} \sigma(s) dW(s) \\ &= \mu(t_k)\Delta t + \sigma(t_k)\Delta W_k + S_k, \end{aligned}$$

where $S_k$ is a remainder term. From this we obtain (3.34)

$$(\Delta X_k)^2 = \mu^2(t_k)(\Delta t)^2 + 2\mu(t_k)\sigma(t_k)\Delta t \Delta W_k + \sigma^2(t_k)(\Delta W_k)^2 + P_k.$$

(3.35)

where $P_k$ is a remainder term. If we now substitute (3.33)–(3.35) into (3.32) we obtain, in shorthand notation,

$$f(t,X(t)) - f(0,X(0)) = I_1 + I_2 + I_3 + \frac{1}{2} I_4 + \frac{1}{2} K_1 + K_2 + R,$$

where

$$\begin{array}{llll} I_1 & = \Sigma_k f_t(t_k)\Delta t, I_2 & & = \Sigma_k f_x(t_k)\mu(t_k)\Delta t, \\ I_3 & = \Sigma_k f_x(t_k)\sigma(t_k)\Delta W_k, I_4 & & = \Sigma_k f_{xx}(t_k)\sigma^2(t_k)(\Delta W_k)^2, \\ K_1 & = \Sigma_k f_{xx}(t_k)\mu^2(t_k)(\Delta t)^2, K_2 & & = \Sigma_k f_{xx}(t_k)\mu(t_k)\sigma(t_k)\Delta t \Delta W_k, \\ R & = \Sigma_k \{Q_k + S_k + P_k\}. & & \end{array}$$

Letting $n \to \infty$ we have, more or less by definition,

$$I_1 \longrightarrow \int_0^t f_t(s, X(s))ds, \quad I_2 \longrightarrow \int_0^t f_x(s, X(s))\mu(s)ds,$$
$$I_3 \longrightarrow \int_0^t f_x(s, X(s))\sigma(s)dW(s).$$

Very much as when we proved earlier that $\sum (\Delta W_k)^2 \to t$, it is possible to show that

$$I_4 \to \int_0^t f_{xx}(s, X(s))\sigma^2(s)ds,$$

and it is fairly easy to show that $K_1$ and $K_2$ converge to zero. The really hard part is to show that the term $R$, which is a large sum of individual remainder terms, also converges to zero. This can, however, also be done and the proof is finished.

# 3.6 Examples

In order to illustrate the use of the Itô formula we now give some examples. All these examples are quite simple, and the results could have been obtained as well by using standard techniques from elementary probability theory. The full force of the Itô calculus will be seen in the following chapters.

The first two examples illustrate a useful technique for computing expected values in situations involving Wiener processes. Since arbitrage pricing to a large extent consists of precisely the computation of certain expected values this technique will be used repeatedly in the sequel.

Suppose that we want to compute the expected value $E[Y]$ where $Y$ is some stochastic variable. Schematically we will then proceed as follows.

1. Try to write $Y$ as $Y=Z(t_0)$ where $t_0$ is some point in time and $Z$ is a stochastic process having an Itô differential.
2. Use the Itô formula to compute $dZ$ as, for example,

$$\begin{aligned} dZ(t) &= \mu(t)dt + \sigma(t)dW(t), \\ Z(0) &= z_0. \end{aligned}$$

3. Write this expression in integrated form as

$$Z(t) = z_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s).$$

4. Take expected values. Using Proposition 3.4 we see that the $dW$-integral will vanish. For the $ds$-integral we may move the expectation operator inside the integral sign (an integral is "just" a sum), and we thus have

$$E[Z(t)] = z_0 + \int_0^t E[\mu(s)]ds.$$

Now two cases can occur.

(a) We may, by skill or pure luck, be able to calculate the expected value $E[\mu(s)]$ explicitly. Then we only have to compute an ordinary Riemann integral to obtain $E[Z(t)]$, and thus to read off $E[Y] = E[Z(t_0)]$.
(b) If we cannot compute $E[\mu(s)]$ directly we have a harder problem, but in some cases we may convert our problem to that of solving an ordinary differential equation (ODE).

**Example 3.12** Compute $E[W^4(t)]$.

**Solution:**

Define $Z$ by $Z(t) = W^4(t)$. Then we have $Z(t) = f(t,X(t))$ where $X = W$ and $f$ is given by $f(t,x) = x^4$. Thus the stochastic differential of $X$ is trivial, namely $dX = dW$, which, in the notation of the Itô formula (3.28), means that $\mu = 0$ and $\sigma = 1$. Furthermore we have $\frac{\partial f}{\partial t} = 0, \frac{\partial f}{\partial x} = 4x^3$ and $\frac{\partial^2 f}{\partial x^2} = 12x^2$. Thus the Itô formula gives us

$$\begin{aligned} dZ(t) &= 6W^2(t)dt + 4W^3(t)dW(t), \\ Z(0) &= 0. \end{aligned}$$

Written in integral form this reads

$$Z(t) = 0 + 6\int_0^t W^2(s)ds + 4\int_0^t W^3(s)dW(s).$$

Now we take the expected values of both members of this expression. Then, by Proposition 3.4, the stochastic integral will vanish. Furthermore we may move the expectation operator inside the $ds$-integral, so we obtain

$$E[Z(t)] = 6\int_0^t E[W^2(s)]ds.$$

Now we recall that $E[W^2(s)] = s$, so in the end we have our desired result

$$E[W^4(t)] = E[Z(t)] = 6\int_0^t s ds = 3t^2.$$

**Example 3.13** Compute $E[e^{\alpha W(t)}]$.

**Solution:**

Define $Z$ by $Z(t) = e^{\alpha W(t)}$. The Itô formula gives us

$$dZ(t) = \frac{1}{2}\alpha^2 e^{\alpha W(t)}dt + \alpha e^{\alpha W(t)}dW(t),$$

so we see that $Z$ satisfies the **stochastic differential equation** (SDE)

$$\begin{aligned} dZ(t) &= \frac{1}{2}\alpha^2 Z(t)dt + \alpha Z(t)dW(t), \\ Z(0) &= 1. \end{aligned}$$

In integral form this reads

$$Z(t) = 1 + \frac{1}{2}\alpha^2 \int_0^t Z(s)(ds) + \alpha \int_0^t Z(s)dW(s).$$

Taking expected values will make the stochastic integral vanish. After moving the expectation within the integral sign in the *ds*-integral and defining *m* by $m(t) = E\,[Z(t)]$ we obtain the equation

$$m(t) = 1 + \frac{1}{2}\alpha^2 \int_0^t m(s)(ds).$$

This is an integral equation, but if we take the *t*-derivative we obtain the ODE

$$\begin{cases} \dot{m}(t) = \frac{\alpha^2}{2} m(t), \\ m(0) = 1. \end{cases}$$

Solving this standard equation gives us the answer

$$E\left[e^{\alpha W(t)}\right] = E[Z(t)] = m(t) = e^{\alpha^2 t/2}.$$

It is natural to ask whether one can "compute" (in some sense) the value of a stochastic integral. This is a fairly vague question, but regardless of how it is interpreted, the answer is generally no. There are just a few examples where the stochastic integral can be computed in a fairly explicit way. Here is the most famous one.

**Example 3.14** Compute

$$\int_0^t W(s)dW(s).$$

**Solution:**

A natural guess is perhaps that $\int_0^t W(s)dW(s) = \frac{W^2(t)}{2}$. Since Itô calculus does not coincide with ordinary calculus this guess cannot possibly be true, but nevertheless it seems natural to start by investigating the process $Z(t) = W^2(t)$. Using the Itô formula on the function $f(t,x) = x^2$ and with $X = W$ we get

$$dZ(t) = dt + 2W(t)dW(t).$$

In integrated form this reads

$$W^2(t) = t + 2\int_0^t W(s)dW(s),$$

so we get our answer

$$\int_0^t W(s)dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}$$

We end with a useful lemma.

**Lemma 3.15** *Let* σ (*t*) *be a given* **deterministic** *function of time and define the process X by*

$$X(t) = \int_0^t \sigma(s)dW(s). \tag{3.36}$$

*Then X*(*t*) *has a normal distribution with zero mean, and variance given by*

$$Var[X(t)] = \int_0^t \sigma^2(s)ds.$$

This is of course an expected result because the integral is "just" a linear combination of the normally distributed Wiener increments with deterministic coefficients. See the exercises for a hint of the proof.

# 3.7 The Multidimensional Itô Formula

Let us now consider a vector process $X = (X_1, \ldots, X_n)^*$, where the component $X_i$ has a stochastic differential of the form

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)$$

and $W_1, \ldots, W_d$ are $d$ **independent** Wiener processes.

Defining the drift vector μ by

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

the $d$-dimensional vector Wiener process $W$ by

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_d \end{bmatrix},$$

and the $n \times d$-dimensional **diffusion matrix** σ by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nd} \end{bmatrix}.$$

we may write the $X$-dynamics as

$$dX(t) = \mu(t)dt + \sigma(t)dW(t).$$

Let us furthermore define the process $Z$ by

$$Z(t) = f(t, X(t)),$$

where $f : R_+ \times R^n \to R$ is a $C^{1,2}$ mapping. Then, using arguments as above, it can be shown that the stochastic differential $df$ is given by

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j. \tag{3.37}$$

with the extended multiplication rule (see the exercises)

$$(dW_i)(dW_j) = 0, \text{ for } \quad i \neq j.$$

Written out in full (see the exercises) this gives us the following result.

**Theorem 3.16 (Itô's formula)** *Let the n-dimensional process X have dynamics given by*

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

*with notation as above. Then the following hold.*

- *The process $f(t, X(t))$ has a stochastic differential given by*

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\sigma_i dW.$$

*Here the row vector $\sigma_i$ is the ith row of the matrix $\sigma$, i.e.*

$$\sigma_i = [\sigma_{i1}, \dots, \sigma_{id}].$$

*and the matrix C is defined by*

$$C = \sigma\sigma^*,$$

*where * denotes transpose.*

- *Alternatively, the differential is given by the formula*

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j,$$

*with the formal multiplication table*

$$\begin{cases} (dt)^2 &= 0, \\ dt \cdot dW &= 0, \\ (dW_i)^2 &= dt, i = 1, \ldots, d, \\ dW_i \cdot dW_j &= 0, i \neq j. \end{cases}$$

**Remark 3.7.1 (Itô's formula)** The Itô formula can also be written as

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} tr[\sigma^* H \sigma] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW,$$

where $H$ denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

and $tr$ denotes the **trace** of a matrix. The trace is defined, for any square matrix $A$, as the sum of the diagonal elements, i.e.

$$trA = \sum_i A_{ii}.$$

See the exercises for details.

# 3.8 Correlated Wiener Processes

Up to this point we have only considered independent Wiener processes, but sometimes it is convenient to build models based upon Wiener processes which are correlated. In order to define such objects, let us therefore consider $d$ independent standard (i.e. unit variance) Wiener processes $\bar{W}_1, \ldots, \bar{W}_d$. Let furthermore a (deterministic and constant) matrix

$$\delta = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1d} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nd} \end{bmatrix}$$

be given, and consider the $n$-dimensional processes $W$, defined by

$$W = \delta \bar{W}$$

where

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_n \end{bmatrix}.$$

In other words

$$W_i = \sum_{j=1}^{d} \delta_{ij} \overline{W}_j, i = 1, \ldots, n.$$

Let us now **assume that the rows of δ have unit length** , i.e.

$$\|\delta_i\| = 1, \; i = 1, \ldots, n,$$

where the Euclidean norm is defined as usual by

$$\|x\| = \sqrt{\sum_{i=1}^{d} x_i^2}.$$

Then it is easy to see (how?) that each of the components $W_1, \ldots, W_n$ separately are standard (i.e. unit variance) Wiener processes. Let us now define the (instantaneous) **correlation matrix** ϱ of *W* by

$$\rho_{ij} dt = Cov[dW_i, dW_j].$$

We then obtain

$$\begin{aligned} \rho_{ij} dt &= E[dW_i \cdot dW_j] - E[dW_i] \cdot E[dW_j] = E[dW_i \cdot dW_j] \\ &= E\Big[\sum_{k=1}^{d} \delta_{ik} d\overline{W}_k \cdot \sum_{l=1}^{d} \delta_{jl} d\overline{W}_l\Big] = \sum_{kl} \delta_{ik} \delta_{jl} E[d\overline{W}_k \cdot d\overline{W}_l] \\ &= \sum_{k=1}^{d} \delta_{ik} \delta_{jk} = \delta_i \delta_j^*, \end{aligned}$$

i.e.

$$\rho = \delta \delta^*.$$

**Definition 3.17** *The process W, constructed as above, is called a vector of* **correlated** *Wiener processes, with* **correlation matrix** ϱ.

Using this definition we have the following Itô formula for correlated Wiener processes.

**Proposition 3.18 (Itô's formula)** *Take a vector Wiener process* $W = (W_1, \ldots, W_n)$ *with correlation matrix* ϱ *as given, and assume that the vector process* $X = (X_1, \ldots, X_k)^*$ *has a stochastic differential. Then the following hold.*

- *For any $C^{1,2}$ function $f$, the stochastic differential of the process $f(t, X(t))$ is given by*

$$df(t, X(t)) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}dX_i dX_j,$$

*with the formal multiplication table*

$$\begin{cases} (dt)^2 &= 0, \\ dt \cdot dW_i &= 0, \; i = 1, \ldots, n, \\ dW_i \cdot dW_j &= \rho_{ij} dt. \end{cases}$$

- *If, in particular, $k = n$ and $dX$ has the structure*

$$dX_i = \mu_i dt + \sigma_i dW_i, \; i = 1, \ldots, n,$$

*where $\mu_1, \ldots, \mu_n$ and $\sigma_1, \ldots, \sigma_n$ are scalar processes, then the stochastic differential of the process $f(t, X(t))$ is given by*

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial x_i} dW_i.$$

We end this section by showing how it is possible to translate between the two formalisms above. Suppose therefore that the $n$-dimensional process $X$ has a stochastic differential of the form

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \tag{3.38}$$

i.e.

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t), \; i = 1, \ldots, n. \tag{3.39}$$

Thus the drift vector process $\mu$ is given by

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

and the diffusion matrix process $\sigma$ by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nd} \end{bmatrix}.$$

$W$ is assumed to be $d$-dimensional standard vector Wiener process (i.e. with independent components) of the form

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_d \end{bmatrix}.$$

The system (3.38) can also be written as

$$dX_i(t) = \mu_i(t)dt + \sigma_i(t)dW(t),$$

where, as usual, $\sigma_i$ is the ith row of the matrix σ. Let us now define $n$ new scalar Wiener processes $\overline{W}_1, \ldots, \overline{W}_n$ by

$$\overline{W}_i = \frac{1}{\|\sigma_i\|}\sigma_i W, \; i = 1 \ldots n.$$

We can then write the $X$-dynamics as

$$dX_i(t) = \mu_i(t)dt + \|\sigma_i(t)\| d\overline{W}_i(t), \; i = 1, \ldots, n. \tag{3.40}$$

As is easily seen, each $\overline{W}_i$ is a standard scalar Wiener process, but $\overline{W}_1, \ldots, \overline{W}_d$ are of course correlated. The local correlation is easily calculated as

$$\rho_{ij} = E[d\overline{W}_i d\overline{W}_j] = \frac{1}{\|\sigma_i\| \cdot \|\sigma_j\|} \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk} = \frac{\sigma_i \sigma_j^*}{\|\sigma_i\| \cdot \|\sigma_j\|}$$

Summing up we have the following result.

**Proposition 3.19** *The system*

$$dX_i(t) = \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_i(t), \; i = 1, \ldots, n,$$

*where* $W_1, \ldots, W_d$ *are independent, may equivalently be written as* (3.41)

$$dX_i(t) = \mu_i(t)dt + \delta_i(t)d\overline{W}_i(t), \; i = 1, \ldots, n, \tag{3.42}$$

*where* $\overline{W}_1, \ldots, \overline{W}_d$ *have the local correlation matrix* ϱ. *The connections between* (3.41) *and* (3.42) *are given by the following expressions.*

$$\begin{cases} \overline{W}_i = \frac{1}{\|\sigma_i\|}\sigma_i W, & i = 1, \ldots, n, \\ \delta_i = \|\sigma_i\|, & i = 1, \ldots, n, \\ \rho_{ij} = \frac{\sigma_i \sigma_j^*}{\|\sigma_i\| \cdot \|\sigma_j\|}, & i,j = 1, \ldots, n. \end{cases}$$

# 3.9 Exercises

**Exercise 3.1** Compute the stochastic differential $dZ$ when

(a) $Z(t) = e^{\alpha t}$,
(b) $Z(t) = \int_0^t g(s)dW(s)$, where $g$ is an adapted stochastic process.
(c) $Z(t) = e^{\alpha W(t)}$
(d) $Z(t) = e^{\alpha X(t)}$, where $X$ has the stochastic differential

$$dX(t) = \mu dt + \sigma dW(t)$$

($\mu$ and $\sigma$ are constants).
(e) $Z(t) = X^2(t)$, where $X$ has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

**Exercise 3.2** Compute the stochastic differential for $Z$ when $Z(t) = \frac{1}{X(t)}$ and $X$ has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

By using the definition $Z = X^{-1}$ you can in fact express the right hand side of $dZ$ entirely in terms of $Z$ itself (rather than in terms of $X$). Thus $Z$ satisfies a stochastic differential equation. Which one?

**Exercise 3.3** Let $\sigma(t)$ be a given deterministic function of time and define the process $X$ by

$$X(t) = \int_0^t \sigma(s)dW(s). \tag{3.43}$$

Use the technique described in Example 3.13 in order to show that the characteristic function of $X(t)$ (for a fixed $t$) is given by

$$E\left[e^{iuX(t)}\right] = exp\left\{-\frac{u^2}{2}\int_0^t \sigma^2(s)ds\right\}, u \in R, \tag{3.44}$$

thus showing that $X(t)$ is normally distibuted with zero mean and a variance given by

$$Var[X(t)] = \int_0^t \sigma^2(s)ds.$$

**Exercise 3.4** Suppose that $X$ has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma(t)dW(t),$$

where $\alpha$ is a real number whereas $\sigma(t)$ is any stochastic process. Use the technique in Example 3.13 in order to determine the function $m(t) = E[X(t)]$.

**Exercise 3.5** Suppose that the process $X$ has a stochastic differential

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

and that $\mu(t) \geq 0$ with probability one for all $t$. Show that this implies that $X$ is a submartingale.

**Exercise 3.6** A function $h(x_1, \ldots, x_n)$ is said to be **harmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

It is **subharmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Let $W_1, \ldots, W_n$ be independent standard Wiener processes, and define the process $X$ by $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that $X$ is a martingale (submartingale) if $h$ is harmonic (subharmonic).

**Exercise 3.7** The object of this exercise is to give an argument for the formal identity

$$dW_1 \cdot dW_2 = 0,$$

when $W_1$ and $W_2$ are independent Wiener processes. Let us therefore fix a time $t$, and divide the interval $[0,t]$ into equidistant points $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{i}{n} \cdot t$. We use the notation

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \quad i = 1,2.$$

Now define $Q_n$ by

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$

Show that $Q_n \rightarrow 0$ in $L^2$, i.e. show that

$$\begin{aligned} E[Q_n] &= 0, \\ Var[Q_n] &\longrightarrow 0. \end{aligned}$$

**Exercise 3.8** Let $X$ and $Y$ be given as the solutions to the following system of stochastic differential equations.

$$\begin{aligned} dX &= \alpha X dt - Y dW, \; X(0) = x_0, \\ dY &= \alpha Y dt + X dW, \; Y(0) = y_0. \end{aligned}$$

Note that the initial values $x_0, y_0$ are deterministic constants.

(a) Prove that the process $R$ defined by $R(t) = X^2(t) + Y^2(t)$ is deterministic.
(b) Compute $E[X(t)]$.

**Exercise 3.9** For a $n \times n$ matrix $A$, the **trace** of $A$ is defined as

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) If $B$ is $n \times d$ and $C$ is $d \times n$, then $BC$ is $n \times n$. Show that

$$tr(BC) = \sum_{ij} B_{ij} C_{ji}$$

(b) With assumptions as above, show that

$$tr(BC) = tr(CB).$$

(c) Show that the Itô formula in Theorem 3.16 can be written

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} tr[\sigma^* H \sigma] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW_i$$

where $H$ denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

**Exercise 3.10** Prove all claims in Section 3.8.

## 3.10 Notes

As a (far reaching) introduction to stochastic calculus and its applications, Øksendal (1995) is in a class of its own. Standard references on a more advanced level are Karatzas and Shreve (1988), and Revuz and Yor (1991). The theory of stochastic integration can be extended from the Wiener framework to allow for semimartingales as integrators, and a classic in this field is Meyer (1976). Standard references are Jacod and Shiryaev (1987), Elliott (1982), and Dellacherie and Meyer (1972). An alternative to the classic approach to semimartingale integration theory is presented in Protter (1990).

# 4 Differential Equations

## 4.1 Stochastic Differential Equations

Let $M(n,d)$ denote the class of $n \times d$ matrices, and consider as given the following objects.

- A $d$-dimensional (column-vector) Wiener process $W$.
- A (column-vector valued) function $\mu$: $R_+ \times R^n \rightarrow R^n$.
- A function $\sigma$: $R_+ \times R^n \rightarrow M(n,d)$.
- A real (column) vector $x_0 \in R^n$.

We now want to investigate whether there exists a stochastic process $X$ which satisfies the **stochastic differential equation** (SDE)

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \tag{4.1}$$

$$X_0 = x_0. \tag{4.2}$$

To be more precise we want to find a process $X$ satisfying the integral equation

$$X_t = x_0 + \int_0^t \mu(s,X_s) \quad ds + \int_0^t \sigma(s,X_s) \quad dW_s, \text{ for all } t \geq 0. \tag{4.3}$$

The standard method for proving the existence of a solution to the SDE above is to construct an iteration scheme of Cauchy-Picard type. The Idea is to define a sequence of processes $X^0$, $X^1$, $X^2$, . . . according to the recursive definition

$$X_t^0 = x_0, \tag{4.4}$$

$$X_t^{n+1} = x_0 + \int_0^t \mu(s,X_s^n) \quad ds + \int_0^t \sigma(s,X_s^n) \quad dW_s. \tag{4.5}$$

Having done this one expects that the sequence $\{X^n\}_{n=1}^{\infty}$ will converge to some limiting process $X$, and that this $X$ is a solution to the SDE. This construction can in fact be carried out, but as the proof requires some rather hard inequalities we only give the result.

**Proposition 4.1** *Suppose that there exists a constant K such that the following conditions are satisfied for all x, y and t.*

$$\|\mu(t,x) - \mu(t,y)\| \leq K\|x - y\|, \tag{4.6}$$

$$\|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|, \tag{4.7}$$

$$\|\mu(t,x)\| + \|\sigma(t,x)\| \le K(1 + \|x\|). \tag{4.8}$$

*Then there exists a unique solution to the SDE (4.1)-(4.2). The solution has the properties*

1. *$X$ is $\mathcal{F}_t^W$-adapted.*
2. *$X$ has continuous trajectories.*
3. *$X$ is a Markov process.*
4. *There exists a constant $C$ such that*

$$E\left[\|X_t\|^2\right] \le Ce^{Ct}\left(1 + \|x_0\|^2\right) \tag{4.9}$$

The fact that the solution $X$ is $\mathcal{F}_t^W$-adapted means that for each fixed $t$ the process value $X_t$ is a functional of the Wiener trajectory on the interval $[0, t]$, and in this way an SDE induces a transformation of the space $C[0, \infty)$ into itself, where a Wiener trajectory $W_.(\omega)$ is mapped to the corresponding solution trajectory $X_.(\omega)$. Generically this transformation, which takes a Wiener trajectory into the corresponding $X$-trajectory, is enormously complicated and it is extremely rare that one can "solve" an SDE in some "explicit" manner. There are, however, a few nontrivial interesting cases where it is possible to solve an SDE, and the most important example for us is the equation below, describing the so called geometric Brownian motion (GBM).

## 4.2 Geometric Brownian Motion

Geometric Brownian motion will be one of our fundamental building blocks for the modeling of asset prices, and it also turns up naturally in many other places. The equation is one of two natural generalizations of the simplest linear ODE and looks as follows.

**Geometric Brownian motion:**

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \tag{4.10}$$

$$X_0 = x_0. \tag{4.11}$$

Written in a slightly sloppy form we can write the equation as

$$\dot{X}_t = \left(\alpha + \sigma \dot{W}_t\right) X_t$$

where $\dot{W}$ is "white noise", i.e. the (formal) time derivative of the Wiener process. Thus we see that GBM can be viewed as a linear ODE, with a stochastic coefficient driven by white noise. See Fig. 4.1, for a computer simulation of GBM with $\alpha = 1$, $\sigma = 0.2$ and $X(0) = 1$. The smooth line is the graph of the expected value function $E[X_t] = 1 \cdot e^{\alpha t}$. For small values of $\sigma$, the trajectory will (at least initially) stay fairly close to the expected value function, whereas a large value of $\sigma$ will give rise to large random deviations. This can clearly be seen when we

Fig. 4.1 Geometric Brownian Motion: $\alpha = 1$, $\sigma = 0.2$

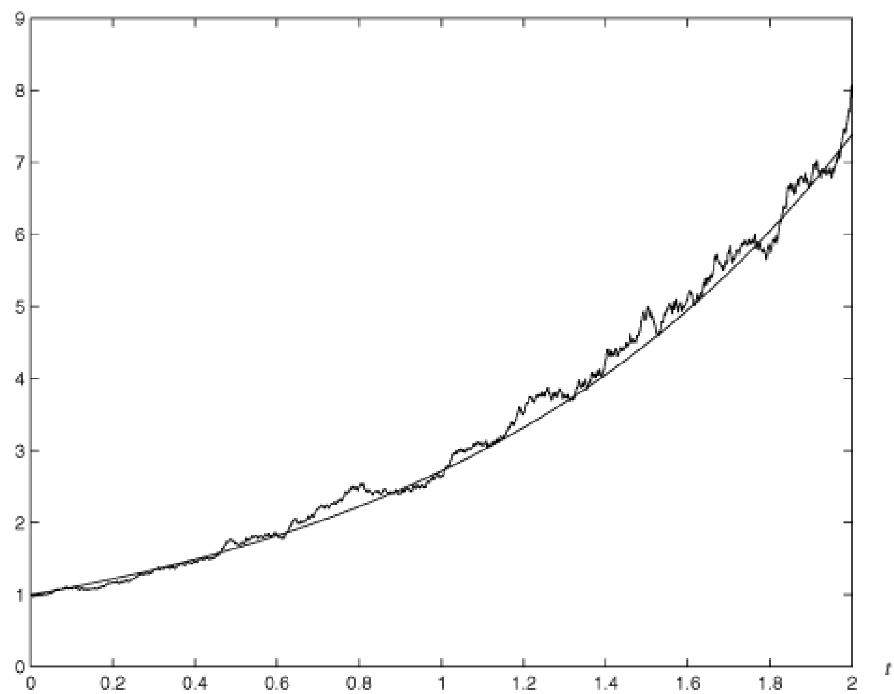
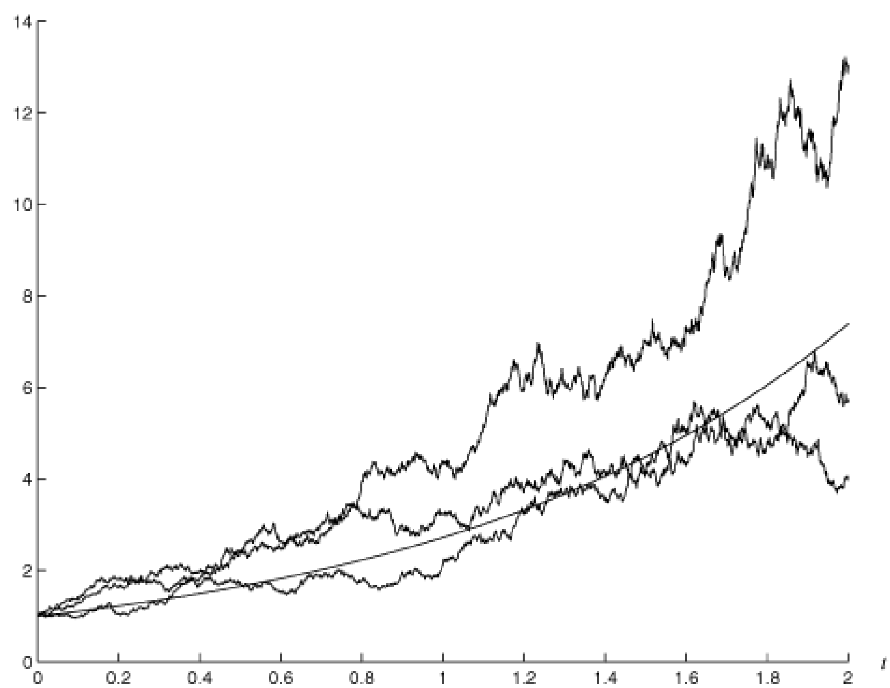



Fig. 4.2 Geometric Brownian Motion: $\alpha = 1$, $\sigma = 0.4$



compare the simulated trajectory in Fig. 4.1 to the three simulated trajectories in Fig. 4.2 where we have $\sigma = 0.4$.

Inspired by the fact that the solution to the corresponding deterministic linear equation is an exponential function of time we are led to investigate the process $Z$, defined by $Z_t = \ln X_t$, where we **assume** that $X$ is a solution and that $X$ is strictly positive (see below). The Itô formula gives us

$$\begin{aligned} dZ &= \frac{1}{X}dX + \frac{1}{2}\left\{-\frac{1}{X^2}\right\}[dX]^2 \\ &= \frac{1}{X}\{\alpha X dt + \sigma X dW\} + \frac{1}{2}\left\{-\frac{1}{X^2}\right\}\sigma^2 X^2 dt \\ &= \{\alpha dt + \sigma dW\} - \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Thus we have the equation

$$\begin{aligned} dZ_t &= \left(\alpha - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t, \\ Z_0 &= \ln x_0. \end{aligned}$$

This equation, however, is extremely simple: since the right hand side does not contain $Z$ it can be integrated directly to

$$Z_t = \ln x_0 + \left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t,$$

which means that $X$ is given by

$$X_t = x_0 \cdot \exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}. \tag{4.12}$$

Strictly speaking there is a logical flaw in the reasoning above. In order for $Z$ to be well defined we have to assume that there actually exists a solution $X$ to eqn (4.10) and we also have to assume that the solution is positive. As for the existence, this is covered by Proposition 4.1, but the positivity seems to present a bigger problem. We may actually avoid both these problems by regarding the calculations above as purely heuristic. Instead we **define** the process $X$ by the formula (4.12). Then it is an easy exercise to show that $X$ thus defined actually satisfies the SDE (4.10)–(4.11). Thus we really have proved the first part of the following result, which will be used repeatedly in the sequel. The result about the expected value is an easy exercise, which is left to the reader.

**Proposition 4.2** *The solution to the equation*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t, \tag{4.13}$$

*is given by*

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

$$X(t) = x_0 \cdot \exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}. \tag{4.14}$$

*The expected value is given by*

(4.15)

$$E[X_t] = x_0 e^{\alpha t}.$$

(4.16)

## 4.3 The Linear SDE

In this section we will study the linear SDE, which in the scalar case has the form

$$\begin{cases} dX_t &= aX_t dt + \sigma dW_t, \\ X_0 &= x_0. \end{cases} \tag{4.17}$$

This equation turns up in various physical applications, and we will also meet it below in connection with interest rate theory.

In order to get some feeling for how to solve this equation we recall that the linear ODE

$$\frac{dx_t}{dt} = ax_t + u_t,$$

where $u$ is a deterministic function of time, has the solution

$$x_t = e^{at}x_0 + \int_0^t e^{a(t-s)}u_s ds. \tag{4.18}$$

If we, for a moment, reason heuristically, then it is tempting to formally divide eqn (4.17) by $dt$. This would (formally) give us

$$\frac{dX_t}{dt} = aX_t + \sigma\frac{dW_t}{dt},$$

and, by analogy with the ODE above, one is led to conjecture the formal solution

$$X_t = e^{at}X_0 + \sigma\int_0^t e^{a(t-s)}\frac{dW_s}{ds}ds = e^{at}X_0 + \sigma\int_0^t e^{a(t-s)}dW_s.$$

Generally speaking, tricks like this will not work, since the solution of the ODE is based on ordinary calculus, whereas we have to use Itô calculus when dealing with SDEs. In this case, however, we have a linear structure, which means that the second order term in the Itô formula does not come into play. Thus the solution of the linear SDE is indeed given by the heuristically derived formula above. We formulate the result for a slightly more general situation, where we allow $X$ to be vector-valued.

**Proposition 4.3** *Consider the n-dimensional linear SDE*

$$\begin{cases} dX_t &= (AX_t + b_t)dt + \sigma_t dW_t, \\ X_0 &= x_0 \end{cases} \tag{4.19}$$

*where $A$ is an $n \times n$ matrix, $b$ is an $R^n$-valued deterministic function (in column vector form), σ is a deterministic (row vector) function taking values in $R^d$, and $W$ and d-dimensional Wiener process. The solution of this equation is given by*

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}b_s ds + \int_0^t e^{A(t-s)}\sigma_s dW_s.$$

*Here we have used the matrix exponential $e^{At}$, defined by* (4.20)

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k.$$

**Proof** Defining the process $X$ by (4.20) and using the Itô formula, it is easily seen that $X$ satisfies the SDE (4.19). See the exercises for some details.

In the exercises you will find results about the moments of $X_t$ as well as details about the matrix exponential.

## 4.4 The Infinitesimal Operator

Consider, as in Section 4.1, the *n*-dimensional SDE

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t.$$

(4.21)

Through the Itô formula, the process above is closely connected to a partial differential operator $\boldsymbol{A}$, defined below. The next two sections are devoted to investigating the connections between, on the one hand, the analytical properties of the operator $\boldsymbol{A}$, and on the other hand the probabilistic properties of the process $X$ above.

**Definition 4.4** *Given the SDE in* (*4.21*), *the partial differential operator $\boldsymbol{A}$, referred to as the* **infinitesimal operator** *of X, is defined, for any function h(x) with $h \in C^2(R^n)$, by*

$$Ah(t,x) = \sum_{i=1}^{n} \mu_i(t,x) \frac{\partial h}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x) \frac{\partial^2 h}{\partial x_i \partial x_j}(x),$$

*where as before*

$$C(t,x) = \sigma(t,x)\sigma^*(t,x).$$

This operator is also known as the **Dynkin operator** , the **Itô operator** , or the **Kolmogorov backward operator** . We note that, in terms of the infinitesimal generator, the Itô formula takes the form

$$df(t,X_t) = \left\{\frac{\partial f}{\partial t} + \Box f\right\}dt + [\nabla_x f] \circ dW_t$$

where the gradient $\nabla_x$ is defined for $h \in C^1(R^n)$ as

$$\nabla_x h = \left[\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right].$$

# 4.5 Partial Differential Equations

In this section we will explore the intimate connection which exists between stochastic differential equations and certain parabolic partial differential equations. Consider for example the following so called **Cauchy problem** .

We are given three scalar functions $\mu\ (t,x)$, $\sigma\ (t,x)$ and $\Phi\ (x)$. Our task is to find a function $F$ which satisfies the following **boundary value problem** on $[0,T] \times R$.

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) = 0, \tag{4.22}$$

$$F(T,x) = \Phi(x). \tag{4.23}$$

Now, instead of attacking this problem using purely analytical tools, we will produce a so called **stochastic representation formula** , which gives the solution to (4.22)–(4.23) in terms of the solution to an SDE which is associated to (4.22)–(4.23) in a natural way. Thus we assume that there actually exists a solution $F$ to (4.22)–(4.23). Let us now fix a point in time $t$ and a point in space $x$. Having fixed these we **define** the stochastic process $X$ on the time interval $[t,T]$ as the solution to the SDE

$$dX_s = \mu(s,X_s)\ ds\ + \sigma(s,X_s)\,\mathrm{d}W_s, \tag{4.24}$$

$$X_t = x,$$

and the point is that the infinitesimal generator $A$ for this process is given by (4.25)

$$A = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2},$$

which is exactly the operator apearing in the PDE above. Thus we may write the boundary value problem as

$$\frac{\partial F}{\partial t}(t,x) + AF(t,x) = 0, \tag{4.26}$$

$$F(T,x) = \Phi(x).$$

Applying the Itô formula to the process $F(s,X(s))$ gives us

(4.27)

$$\begin{aligned} F(T,X_T) &= F(t,X_t) \\ &+ \int_t^T \left\{ \frac{\partial F}{\partial t}(s,X_s) + AF(s,X_s) \right\} ds \\ &+ \int_t^T \sigma(s,X_s) \frac{\partial F}{\partial x}(s,X_s) dW_s. \end{aligned}$$

(4.28)

Since, by Assumption, $F$ actually satisfies eqn (4.26), the time integral above will vanish. If furthermore the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is sufficiently integrable and we take expected values, the stochastic integral will also vanish. The initial value $X_t = x$ and the boundary Condition $F(T,x) = \Phi(x)$ will eventually leave us with the formula

$$F(t,x) = E_{t,x}[\Phi(X_T)],$$

where we have indexed the expectation operator in order to emphasize that the expected value is to be taken given the initial value $X_t = x$. Thus we have proved the following result, which is known as the **Feynman-Kač stochastic representation formula** .

**Proposition 4.5 (Feynman-Kač)** *Assume that F is a solution to the boundary value problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

*Assume furthermore that the process*

$$\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$$

*is in* $\mathcal{L}^2$ *(see Definition 3.3), where X is defined below. Then F has the representation*

$$F(t,x) = E_{t,x}[\Phi(X_T)],$$

*where X satisfies the SDE*

(4.29)

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s,$$

$$X_t = x. \quad (4.30)$$

(4.31)

Note that we need the integrability Assumption $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s) \in \mathcal{L}^2$ in order to guarantee that the expected value of the stochastic integral in (4.28) equals zero. In fact the generic situation is that a boundary value problem of the type above—a so called **parabolic** problem—will have infinitely many solutions, (see John 1982). It will, however, only have one "nice" solution, the others being rather "wild", and the proposition above will only give us the "nice" solution.

We may also consider the closely related boundary value problem

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + rF(t,x) = 0, \quad (4.32)$$

$$F(T,x) = \Phi(x), \quad (4.33)$$

where $r$ is a given real number. Equations of this type appear over and over again in the study of pricing problems for financial derivatives. Inspired by the ODE technique of integrating factors we are led to multiply the entire eqn (4.32) by the factor $e^{rs}$, and if we then consider the process $Z(s) = e^{rs}F(s,X(s))$, where $X$ as before is defined by (4.30)–(4.31), we obtain the following result.

**Proposition 4.6 (Feynman-Kač)** *Assume that F is a solution to the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + rF(t,x) = 0, \quad (4.34)$$

$$F(T,x) = \Phi(x). \quad (4.35)$$

*Assume furthermore that the process* $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ *is in* $\pounds^2$, *where X is defined below. Then F has the representation*

$$F(t,x) = e^{r(T-t)}E_{t,x}[\Phi(X_T)], \quad (4.36)$$

*where X satisfies the SDE*

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \quad (4.37)$$

$$X_t = x. \quad (4.38)$$

**Example 4.7** Solve the PDE

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

$$F(T,x) = x^2,$$

where σ is a constant.

**Solution:**

From Proposition 4.5 we immediately have

$$F(t,x) = E_{t,x}\left[X_T^2\right],$$

where

$$dX_s = 0 \cdot ds + \sigma dW_s,$$

$$X_t = x.$$

This equation can easily be solved, and we have

$$X_T = x + \sigma[W_T - W_t],$$

so $X_T$ has the distribution $N[x, \sigma\sqrt{T-t}]$. Thus we have the solution

$$\begin{aligned} F(t,x) &= E\left[X_T^2\right] = Var[X_T] + \{E[X_T]\}^2 \\ &= \sigma^2(T-t) + x^2. \end{aligned}$$

Up to now we have only treated the scalar case, but exactly the same arguments as above will give us the following result.

**Proposition 4.8** *Take as given*

- *A (column-vector valued) function* $\mu: R_+ \times R^n \to R^n$.
- *A function* $C: R_+ \times R^n \to M(n,n)$, *which can be written in the form*

$$C(t,x) = \sigma(t,x)\sigma^*(t,x),$$

  *for some function* $\sigma: R_+ \times R^n \to M(n,d)$.
- *A real valued function* $\Phi: R^n \to R$.
- *A real number* $r$.

*Assume that* $F: R_+ \times R^n \to R$ *is a solution to the boundary value problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial F}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) - rF(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

*Assume furthermore that the process*

$$\sum_{i=1}^{n} \sigma_i(s,X_s)\frac{\partial F}{\partial x_i}(s,X_s)$$

*is in* $\mathcal{L}^2$ *(see Definition 3.3), where* $X$ *is defined below. Then* $F$ *has the representation*

$$F(t,x) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)],$$

*where* $X$ *satisfies the SDE* (4.39)

$$dX_s = \mu(s,X_s)dt + \sigma(s,X_s)dW_s, \tag{4.40}$$

$$X_t = x. \tag{4.41}$$

We end this section with a useful result. Given Lemma 3.9 the proof is easy and left to the reader.

**Proposition 4.9** *Consider as given a vector process X with generator* $\mathcal{A}$, *and a function F(t,x). Then, modulo some integrability conditions, the following hold.*

- *The process F(t,X) is a martingale relative to the filtration* $\mathcal{F}^X$ *if and only if F satisfies the PDE*

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0.$$

- *For every (t,x) and T ≥ t, we have*

$$F(t,x) = E_{t,x}[F(T,X_T)].$$

# 4.6 The Kolmogorov Equations

We will now use the results of the previous section in order to derive some classical results concerning the transition probabilities for the solution to an SDE. The discussion has the nature of an overview, so we allow ourselves some latitude as to technical details.

Suppose that $X$ is a solution to the equation

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t, \tag{4.42}$$

with infinitesimal generator $\mathcal{A}$ given by

$$(\mathcal{A}f)(s,y) = \sum_{i=1}^{n} \mu_i(s,y)\frac{\partial f}{\partial y_i}(s,y) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(s,y)\frac{\partial^2 f}{\partial y_i \partial y_j}(s,y),$$

where as usual

$$C(t,x) = \sigma(t,x)\sigma^*(t,x).$$

Now consider the boundary value problem

$$\begin{aligned}\left(\frac{\partial u}{\partial s} + \mathcal{A}u\right)(s,y) &= 0, \ (s,y) \in (0,T) \times R^n,\\ u(T,y) &= I_B(y), \ y \in R^n,\end{aligned}$$

where $I_B$ is the indicator function of the set $B$. From Proposition 4.8 we immediately have

$$u(s,y) = E_{s,y}[I_B(X_T)] = P(X_T \in B | X_s = y),$$

where $X$ is a solution of (4.42). This argument can also be turned around, and we have thus (more or less) proved the following result.

**Proposition 4.10 (Kolmogorov backward equation)** *Let X be a solution to eqn (4.42). Then the transition probabilities* $P(s,y;t,B) = P(X_t \in B | X(s) = y)$ *are given as the solution to the equation*

$$\left(\frac{\partial P}{\partial s} + \mathcal{A}P\right)(s,y;t,B) = 0, \ (s,y) \in (0,t) \times R^n, \tag{4.43}$$

$$P(t,y;t,B) = I_B(y). \tag{4.44}$$

Using basically the same reasoning one can also prove the following corresponding result for transition densities.

**Proposition 4.11 (Kolmogorov backward equation)** *Let X be a solution to eqn (4.42). Assume that the measure P(s,y;t,dx)* *has a density $p(s,y;t,x)dx$. Then we have*

$$\left(\frac{\partial p}{\partial s} + Ap\right)(s,y;\ t,B) = 0,\ (s,y) \in (0,t) \times R^n,$$

$$p(s,y;\ t,x) \to \delta_x,\ \ as\ \ s \to t. \tag{4.45}$$

(4.46)

The reason that eqns (4.43) and (4.45) are called backward equations is that the differential operator is working on the "backward variables" $(s,y)$. We will now derive a corresponding "forward" equation, where the action of the differential operator is on the "forward" variables $(t,x)$. For simplicity we consider only the scalar case.

We assume that $X$ has a transition density. Let us then fix two points in time $s$ and $T$ with $s < T$. Now consider an arbitrary "test function", i.e. an infinite differentiable function $h(t,x)$ with compact support in the set $(s,T) \times R$. From the Itô formula we have

$$h(T,X_T) = h(s,X_s) + \int_s^T \left(\frac{\partial h}{\partial t} + Ah\right)(t,X_t)dt + \int_s^T \frac{\partial h}{\partial x}(t,X_t)dW_t.$$

Applying the expectation operator $E_{s,y}[\cdot]$, and using the fact that, because of the compact support, $h(T,x) = h(s,x) = 0$, we obtain

$$\int_{-\infty}^{\infty}\int_s^T p(s,y;\ t,x)\left(\frac{\partial}{\partial t} + A\right)h(t,x)dxdt = 0.$$

Partial integration with respect to $t$ (for $\frac{\partial}{\partial t}$) and with respect to $x$ (for $A$) gives us

$$\int_{-\infty}^{\infty}\int_s^T h(t,x)\left(-\frac{\partial}{\partial t} + A^*\right)p(s,y;\ t,x)dxdt = 0,$$

where the adjoint operator $A^*$ is defined by

$$\left(A^* f\right)(t,x) = -\frac{\partial}{\partial x}[\mu(t,x)f(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t,x)f(t,x)\right].$$

Since this equation holds for all test functions we have shown the following result.

**Proposition 4.12 (Kolmogorov forward equation)** *Assume that the solution $X$ of eqn (4.42) has a transition density $p(s,y;t,x)$. Then $p$ will satisfy the* **Kolmogorov forward equation**

$$\frac{\partial}{\partial t}p(s,y;\ t,x) = A^* p(s,y;\ t,x),\ \ (t,x) \in (0,T) \times R,$$

$$p(s,y;\ t,x) \to \delta_y,\ \ as\ \ t \downarrow s. \tag{4.47}$$

(4.48)

This equation is also known as the **Fokker-Planck equation** . The multidimensional version is readily obtained as

$$\frac{\partial p}{\partial t}p(s,y;\ t,x) = A^* p(s,y;\ t,x),$$

where the adjoint operator $A^*$ is defined by

$$\left(A^* f\right)(t,x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_i(t,x) f(t,x)] + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[C_{ij}(t,x) f(t,x)].$$

**Example 4.13** Let us consider a standard Wiener process with constant diffusion coefficient $\sigma$, i.e. the SDE

$$dX_t = \sigma dW_t.$$

The Fokker-Planck equation for this process is

$$\frac{\partial p}{\partial t}(s,y;t,x) = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}(s,y;t,x),$$

and it is easily checked that the solution is given by the Gaussian density

$$p(s,y;t,x) = \frac{1}{\sigma\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}\frac{(x-y)^2}{\sigma^2(t-s)}\right].$$

**Example 4.14** Consider the GBM process

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

The Fokker-Planck equation for this process is

$$\frac{\partial p}{\partial t}(s,y;t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2 x^2 p(s,y;t,x)\right] - \frac{\partial}{\partial x}[\alpha x p(s,y;t,x)],$$

i.e.

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2} + \left(2\sigma^2 - \alpha\right) x \frac{\partial p}{\partial x} + \left(\sigma^2 - \alpha\right) p.$$

A change of variables of the form $x = e^y$ reduces this equation to an equation with constant coefficients, which can be solved by Fourier methods. For us it is perhaps easier to get the transition density directly by solving the SDE above. See the exercises below.

## 4.7 Exercises

**Exercise 4.1** Show that the scalar SDE

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma dW_t, \\ X_0 &= x_0, \end{aligned}$$

has the solution

$$X(t) = e^{\alpha t} \cdot x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW(s), \tag{4.49}$$

by differentiating $X$ as defined by eqn (4.49) and showing that $X$ so defined actually satisfies the SDE.

**Hint:** Write eqn (4.49) as

$$X_t = Y_t + Z_t \cdot R_t,$$

where

$$Y_t = e^{\alpha t} \cdot x_0,$$
$$Z_t = e^{\alpha t} \cdot \sigma,$$
$$R_t = \int_0^t e^{-\alpha s} dW_s,$$

and first compute the differentials $dZ$, $dY$ and $dR$. Then use the multidimensional Itô formula on the function $f(y,z,r) = y + z \cdot r$.

**Exercise 4.2** Let $A$ be an $n \times n$ matrix, and define the matrix exponential $e^A$ by the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series can be shown to converge uniformly.

(a) Show, by taking derivatives under the summation sign, that

$$\frac{de^{At}}{dt} = Ae^{At}.$$

(b) Show that

$$e^0 = I,$$

where 0 denotes the zero matrix, and $I$ denotes the identity matrix.

(c) Convince yourself that if $A$ and $B$ commute, i.e. $AB = BA$, then

$$e^{A+B} = e^A \cdot e^B = e^B \cdot e^A.$$

**Hint:** Write the series expansion in detail.

(d) Show that $e^A$ is invertible for every $A$, and that in fact

$$\left[e^A\right]^{-1} = e^{-A}.$$

(e) Show that for any $A$, $t$ and $s$

$$e^{A(t+s)} = e^{At} \cdot e^{As}$$

(f) Show that

$$\left(e^A\right)^* = e^{A^*}$$

**Exercise 4.3** Use the exercise above to complete the details of the proof of Proposition 4.3.

**Exercise 4.4** Consider again the linear SDE (4.19). Show that the expected value function $m(t) = E[X(t)]$, and the covariance matrix $C(t) = \{Cov\,(X_i(t), X_j(t)\}_{i,j}$ are given by

$$\begin{aligned} m(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds, \\ C(t) &= \int_0^t e^{A(t-s)}\sigma(s)\sigma^*(s)e^{A^*(t-s)}ds, \end{aligned}$$

where * denotes transpose.

**Hint:** Use the explicit solution above, and the fact that

$$C(t) = E\left[X_t X_t^*\right] - m(t)m^*(t).$$

Geometric Brownian motion (GBM) constitutes a class of processes which is closed under a number of nice operations. Here are some examples.

**Exercise 4.5** Suppose that $X$ satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Now define $Y$ by $Y_t = X_t^{\beta}$, where β is a real number. Then $Y$ is also a GBM process. Compute $dY$ and find out which SDE $Y$ satisfies.

**Exercise 4.6** Suppose that $X$ satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and $Y$ satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where $V$ is a Wiener process which is independent of $W$. Define $Z$ by $Z = \frac{X}{Y}$ and derive an SDE for $Z$ by computing $dZ$ and substituting $Z$ for $\frac{X}{Y}$ in the right hand side of $dZ$. If $X$ is nominal income and $Y$ describes inflation then $Z$ describes real income.

**Exercise 4.7** Suppose that $X$ satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and $Y$ satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dW_t.$$

Note that now both $X$ and $Y$ are driven by the same Wiener process $W$. Define $Z$ by $Z = \frac{X}{Y}$ and derive an SDE for $Z$.

**Exercise 4.8** Suppose that $X$ satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and $Y$ satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where $V$ is a Wiener process which is independent of $W$. Define $Z$ by $Z = X \cdot Y$ and derive an SDE for $Z$. If $X$ describes the price process of, for example, IBM in US\$ and $Y$ is the currency rate SEK/US\$ then $Z$ describes the dynamics of the IBM stock expressed in SEK.

**Exercise 4.9** Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0,T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} &= 0, \\ F(T,x) &= ln(x^2). \end{aligned}$$

Here $\mu$ and $\sigma$ are assumed to be known constants.

**Exercise 4.10** Consider the following boundary value problem in the domain $[0,T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + k(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

Here $\mu$, $\sigma$, $k$ and $\Phi$ are assumed to be known functions.

Prove that this problem has the stochastic representation formula

$$F(t,x) = E_{t,x}[\Phi(X_T)] + \int_t^T E_{t,x}[k(s,X_s)]\,ds,$$

where as usual $X$ has the dynamics

$$\begin{aligned} dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \\ X_t &= x. \end{aligned}$$

**Hint:** Define $X$ as above, assume that $F$ actually solves the PDE and consider the process $Z_s = F(s,X_s)$.

**Exercise 4.11** Use the result of the previous exercise in order to solve

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2 \frac{\partial^2 F}{\partial x^2} + x &= 0, \\ F(T,x) &= \ln(x^2). \end{aligned}$$

**Exercise 4.12** Consider the following boundary value problem in the domain $[0,T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + r(t,x)F &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

Here $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$ and $\Phi(x)$ are assumed to be known functions. Prove that this problem has a stochastic representation formula of the form

$$F(t,x) = E_{t,x}\left[\Phi(X_T)e^{{}_t^T r(s,X_s)ds}\right],$$

by considering the process $Z_s = F(s,X_s) \times \exp\left[\int_t^s r(u,X_u)du\right]$ on the time interval $[t,T]$.

**Exercise 4.13** Solve the boundary value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x,y) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\delta^2\frac{\partial^2 F}{\partial y^2}(t,x,y) &= 0, \\ F(T,x,y) &= xy. \end{aligned}$$

**Exercise 4.14** Go through the details in the derivation of the Kolmogorov forward equation.

**Exercise 4.15** Consider the SDE

$$dX_t = \alpha dt + \sigma dW_t,$$

where $\alpha$ and $\sigma$ are constants.

(a) Compute the transition density $p(s,y;t,x)$, by solving the SDE.
(b) Write down the Fokker-Planck equation for the transition density and check the equation is indeed satisfied by your answer in (a).

**Exercise 4.16** Consider the standard GBM

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

and use the representation

$$X_t = X_s \exp\left\{\left[\alpha - \frac{1}{2}\sigma^2\right](t-s) + \sigma[W_t - W_s]\right\}$$

in order to derive the transition density $p(s,y;t,x)$ of GBM. Check that this density satisfies the Fokker-Planck equation in Example 4.14.

## 4.8 Notes

All the results in this chapter are standard and can be found in, for example, Karatzas and Shreve (1988), Revuz and Yor (1991), Øksendal (1995). For an encyclopedic treatment of the probabilistic approach to parabolic PDEs see Doob (1984).

# 5 Portfolio Dynamics

## 5.1 Introduction

Let us consider a financial market consisting of different assets such as stocks, bonds with different maturities, or various kinds of financial derivatives. In this chapter we will take the price dynamics of the various assets as given, and the main objetive is that of deriving the dynamics of (the value of) a so called **self-financing** portfolio. In continuous time this turns out to be a fairly delicate task, so we start by studying a model in discrete time. We will then let the length of the time step tend to zero, thus obtaining the continuous time analogs. It is to be stressed that this entire section is only motivating and heuristic. The formal definitions and the corresponding theory will be given in the next section.

Let us thus study a financial market, where time is divided into periods of length δ, and where trading only takes place at the discrete points in time $n \Delta t$, $n = 0, 1, \ldots$. We consider a fixed period $[t, t + \Delta t)$. This period (where of course $t = n \Delta t$ for some $n$) is henceforth referred to as "period $t$". In the sequel we will assume that all assets are stocks, but this is purely for linguistic convenience.

**Definition 5.1**

$$
\begin{aligned}
N &= \textit{the number of different types of stocks.} \\
h_i(t) &= \textit{number of shares of type } i \textit{ held during the period } [t, t + \Delta t). \\
h(t) &= \textit{the portfolio } [h_1(t), \ldots, h_N(t)] \textit{ held during period } t. \\
c(t) &= \textit{the amount of money spent on consumption per unit time} \\
&\quad \textit{during the period } [t, t + \Delta t) \\
S_i(t) &= \textit{the price of one share of type } i \textit{ during the period } [t, t + \Delta t). \\
V(t) &= \textit{the value of the portfolio } h \textit{ at time } t.
\end{aligned}
$$

The information and the decisions in the model are structured as follows.

- At time $t$, i.e. at the **start** of period $t$, we bring with us an "old" portfolio $h(t - \Delta t) = \{h_i(t - \Delta t), i = 1, \ldots, N\}$ from the previous period $t - \Delta t$.
- At time $t$ we can observe the price vector $S(t) = (S_1(t), \ldots, S_N(t))$.
- At time $t$, after having observed $S(t)$, we choose a new portfolio $h(t)$, to be held during period $t$. At the same time we also choose the consumption rate $c(t)$ for the period $t$. Both $h(t)$ and $c(t)$ are assumed to be constant over the period $t$.

**Remark 5.1.1** Note that, so far, we only consider **nondividend paying** assets. The case of dividend paying assets is slightly more complicated, and since it will only be used in Chapter 11, we omit it from our main discussion. See Section 5.3 below for details.

We will only consider so called **self-financing** portfolio–consumption pairs ($h$,$c$), i.e. portfolios with no exogenous infusion or withdrawal of money (apart of course from the $c$-term). In other words, the purchase of a new portfolio, as well as all consumption, must be financed solely by selling assets already in the portfolio.

To start the analysis we observe that our wealth $V(t)$, i.e. the wealth at the **start** of period $t$, equals the value of the old portfolio $h(t - \Delta t)$. Thus we have

$$V(t) = \sum_{i=1}^{N} h_i(t - \Delta t)S_i(t) = h(t - \Delta t)S(t), \tag{5.1}$$

where we have used the notation

$$xy = \sum_{i=1}^{N} x^i y^i$$

for the inner product in $R^N$. Equation (5.1) simply says that at the beginning of period $t$ our wealth equals what we get if we sell our old portfolio at today's prices. We may now use the proceeds of this sale for two purposes.

- Reinvest in a new portfolio $h(t)$.
- Consume at the rate $c(t)$ over the period $t$.

The cost of the new portfolio $h(t)$, which has to be bought at today's prices, is given by

$$\sum_{i=1}^{N} h_i(t)S_i(t) = h(t)S(t),$$

whereas the cost for the consumption rate $c(t)$ is given by $c(t)\,\Delta\, t$. The budget equation for period $t$ thus reads

$$h(t - \Delta t)S(t) = h(t)S(t) + c(t)\Delta t. \tag{5.2}$$

If we introduce the notation

$$\Delta X(t) = X(t) - X(t - \Delta t),$$

for an arbitrary process $X$, we see that the budget equation (5.2) reads

$$S(t)\Delta h(t) + c(t)\Delta t = 0. \tag{5.3}$$

Since our goal is to obtain the budget equation in continuous time it is now tempting to let $\Delta\, t \to 0$ in eqn (5.3) to obtain the formal expression

$$S(t)dh(t) + c(t)dt = 0.$$

This procedure is, however, **not** correct, and it is important to understand why that is so. The reasons are as follows.

- All stochastic differentials are to be interpreted in the Itô sense.
- The Itô integral $\int g(t)dW(t)$ was defined as the limit of sums of the type

$$\sum g(t_n)[W(t_{n+1}) - W(t_n)],$$

  where it was essential that the $W$-increments were **forward** differences.
- In eqn (5.3) we have a **backward** $h$-difference.

In order to get Itô differentials we thus have to reformulate eqn (5.3). This is done by adding and subtracting the term $S(t-\Delta t)\,\Delta h(t)$ to the left hand side, and the budget equation now reads

$$S(t-\Delta t)\Delta h(t) + \Delta S(t)\Delta h(t) + c(t)\Delta t = 0. \quad (5.4)$$

Now, at last, we may let $\Delta t \rightarrow 0$ in the budget eqn (5.4), giving us

$$S(t)dh(t) + dh(t)dS(t) + c(t)dt = 0. \quad (5.5)$$

Letting $\Delta t \rightarrow 0$ in eqn (5.1) gives us

$$V(t) = h(t)S(t), \quad (5.6)$$

and if we take the Itô differential of this expression we get

$$dV(t) = h(t)dS(t) + S(t)dh(t) + dS(t)dh(t). \quad (5.7)$$

To sum up, eqn (5.7) is the general equation for the dynamics of an arbitrary portfolio, and eqn (5.5) is the budget equation which holds for all self-financing portfolios. Substituting (5.5) into (5.7) thus gives us our desired object, namely the dynamics of (the wealth of) a self-financing portfolio.

$$dV(t) = h(t)dS(t) - c(t)dt. \quad (5.8)$$

In particular we see that in a situation without any consumption we have the following $V$-dynamics.

$$dV(t) = h(t)dS(t). \quad (5.9)$$

**Remark 5.1.2** The natural economic interpretation of eqn (5.9) is of course that in a model without any exogenous income, all change of wealth is due to changes in asset prices. Thus (5.8) and (5.9) seem to be rather self-evident, and one may think that our derivation was rather unneccesary. This is, however, not the case, which we realize if we recall that the stochastic differentials in (5.8) and (5.9) are to be interpreted in the Itô sense, where it is important that the integrator increment $dS(t)$ is a **forward** increment. If we had chosen to define our stochastic integral in some other way, e.g. by using **backward** increments (this can actually be done), the formal **appearance** of (5.8)–(5.9) would have been quite different. The real **content** , on the other hand, would of course have been the same.

## 5.2 Self-Financing Portfolios

Having gone through the derivations of the preceding section there are some natural questions.

1. In which sense ($L^2$, $P$-a.s., etc.) is the limiting procedure of letting $\Delta t \rightarrow 0$ to be interpreted?
2. Equation (5.8) is supposed to be describing the dynamics of a self-financing portfolio in continuous time, but what is "continuous time trading" supposed to mean "in reality"?

The answer to these questions is simply that the preceding reasoning has only been of a motivating nature. We now give a purely mathematical **definition** of the central concepts. The interpretations of the concepts are of course those of the preceding section.

**Definition 5.2** *Let the N-dimensional price process* $\{S(t);t \geq 0\}$ *be given.*

1. it A **portfolio strategy** *(most often simply called a portfolio) is any* $\mathcal{F}_t^S$*-adapted N-dimensional process* $\{h(t);t \geq 0\}$.
2. *The portfolio h is said to be* **Markovian** *if it is of the form*

$$h(t) = h(t,S(t)),$$

*for some function* $h : R_+ \times R^N \rightarrow R^N$.

3. *The* **value process** $V^h$ *corresponding to the portfolio h is given by*

$$V^h(t) = \sum_{i=1}^{N} h_i(t)S_i(t).$$

4. *A* **consumption process** *is any* $\mathcal{F}_t^S$*-adapted one-dimensional process* $\{c(t);t \geq 0\}$. (5.10)
5. *A portfolio–consumption pair* $(h,c)$ *is called* **self-financing** *if the value process* $V^h$ *satisfies the condition*

$$dV^h(t) = \sum_{i=1}^{N} h_i(t)dS_i(t) - c(t)dt,$$

*i.e. if*

$$dV^h(t) = h(t)dS(t) - c(t)dt. \tag{5.11}$$

**Remark 5.2.1** Note that, in general, the portfolio $h(t)$ is allowed to depend upon the entire past price trajectory $\{S(u); u \leq t\}$. In the sequel we will almost exclusively be dealing with Markovian portfolios, i.e. those portfolios for which the value at time $t$ depends only on today's date $t$ and today's value of the price vector $S(t)$.

For computational purposes it is often convenient to describe a portfolio in relative terms instead of in absolute terms as above. In other words, instead of specifying the absolute number of shares held of a certain stock, we specify the relative proportion of the total portfolio value which is invested in the stock.

**Definition 5.3** *For a given portfolio h the corresponding* **relative portfolio** *u is given by*

$$u_i(t) = \frac{h_i S_i(t)}{V^h(t)}, \; i = 1, \ldots, N, \tag{5.12}$$

*where we have*

$$\sum_{i=1}^{N} u_i(t) = 1.$$

The self-financing condition can now easily be given in terms of the relative portfolio.

**Lemma 5.4** *A portfolio–consumption pair* ($h$,$c$) *is self-financing if and only if*

$$dV^h(t) = V^h(t) \sum_{i=1}^{N} u_i(t) \frac{dS_i(t)}{S_i(t)} - c(t)dt. \tag{5.13}$$

In the future we will need the following slightly technical result which roughly says that if a process **looks** as if it is the value process of a self-financing portfolio, then it actually **is** such a value process.

**Lemma 5.5** *Let $c$ be a consumption process, and assume that there exist a scalar process $Z$ and a vector process $q=(q_1, \ldots, q_N)$ such that*

$$dZ(t) = Z(t) \sum_{i=1}^{N} q_i(t) \frac{dS_i(t)}{S_i(t)} - c(t)dt,$$

$$\sum_{i=1}^{N} q_i(t) = 1. \tag{5.14}$$

*Now define a portfolio $h$ by*

$$h_i(t) = \frac{q_i(t)Z(t)}{S_i(t)}. \tag{5.15}$$

(5.16)

*Then the value process $V^h$ is given by $V^h=Z$, the pair* ($h$,$c$) *is self-financing, and the corresponding relative portfolio $u$ is given by $u = q$.*

**Proof** By definition the value process $V^h$ is given by $V^h(t)=h(t)S(t)$, so eqns (5.15) and (5.16) give us

$$V^h(t) = \sum_{i=1}^{N} h_i(t)S_i(t) = \sum_{i=1}^{N} q_i(t)Z(t) = Z(t) \sum_{i=1}^{N} q_i(t) = Z(t). \tag{5.17}$$

Inserting (5.17) into (5.16) we see that the relative portfolio $u$ corresponding to $h$ is given by $u = q$. Inserting (5.17) and (5.16) into (5.14) we obtain

$$dV^h(t) = \sum_{i=1}^{N} h_i(t) dS_i(t) - c(t)dt,$$

which shows that ($h$,$c$) is self-financing.

## 5.3 Dividends

This section is only needed for Chapter 11. We again consider the setup and notation of Section 5.1, with the addition that the assets now may pay dividends.

**Definition 5.6** *We take as given the processes* $D_1(t), \ldots, D_N(t)$, *where* $D_i(t)$ *denotes the* **cumulative dividends** *paid to the holder of one unit of asset i during the interval* $(0,t]$. If $D_i$ *has the structure*

$$dD_i(t) = \delta_i(t)dt,$$

*for some process* $\delta_i$, *then we say that asset i pays a continuous* **dividend yield** .

The dividends paid to the holder of one unit of asset $i$ during $(s,t]$ are thus given by $D_i(t)-D_i(s)$, and in the case of a dividend yield we have

$$D_i(t) = \int_0^t \delta_i(s)ds.$$

We assume that all the dividend processes have stochastic differentials.

We now go on to derive the dynamics of a self-financing portfolio, and as usual we define the value process $V$ by

$$V(t) = h(t)S(t).$$

The difference between the present situation and the nondividend paying case is that the budget equation (5.2) now has to be modified. We have to take into account the fact that the money at our disposal at time $t$ now consists of two terms.

- The value of our old portfolio, as usual given by

$$h(t-\Delta t)S(t).$$

- The dividends earned during the interval $(t-\Delta t, t]$. These are given by

$$\sum_{i=1}^{N} h_i(t-\Delta t)\left[D_i(t) - D_i(t-\Delta t)\right] = h(t-\Delta t)\Delta D(t).$$

The relevant budget equation is thus given by

$$h(t-\Delta t)S(t) + h(t-\Delta t)\Delta D(t) = h(t)S(t) - c(t)\Delta t. \tag{5.18}$$

Going through the same arguments as in Section 5.1 we end up with the following dynamics for a self-financing portfolio

$$dV(t) = \sum_{i=1}^{N} h_i(t)dS_i(t) + \sum_{i=1}^{N} h_i(t)dD_i(t) - c(t)dt,$$

and we write this as a formal definition.

**Definition 5.7**

1. *The* **value process** $V^h$ *is given by*

$$V^h(t) = \sum_{i=1}^{N} h_i(t)S_i(t). \tag{5.19}$$

2. *The (vector valued)* **gain process** $G$ *is defined by*

$$G(t) = S(t) + D(t). \tag{5.20}$$

3. *The portfolio–consumption pair* $(h,c)$ *is called* **self-financing** *if*

$$dV^h(t) = \sum_{i=1}^{N} h_i(t)dG_i(t) - c(t)dt. \tag{5.21}$$

With notation as above we have the following obvious result.

**Lemma 5.8** *In terms of the relative portfolio weights, the dynamics of a self-financing portfolio can be expressed as*

$$dV^h(t) = V(t) \cdot \sum_{i=1}^{N} u_i(t)\frac{dG_i(t)}{S_i(t)} - c(t)dt \tag{5.22}$$

## 5.4 Exercises

**Exercise 5.1** Work out the details in the derivation of the dynamics of a self-financing portfolio in the dividend paying case.

# 6 Arbitrage Pricing

## 6.1 Introduction

In this chapter we will study a special case of the general model set out in the previous chapter. Let us therefore consider a financial market consisting of only two assets: a risk free asset with price process *B*, and a stock with price process *S*. What, then, is a risk free asset?

**Definition 6.1** *The price process B is the price of a* **risk free** *asset if it has the dynamics*

$$dB(t) = r(t)B(t)dt, \tag{6.1}$$

*where r is any adapted process.*

The defining property of a risk free asset is thus that it has no driving *dW*-term. We see that we also can write the *B*-dynamics as

$$\frac{dB(t)}{dt} = r(t)B(t),$$

so the *B*-process is given by the expression

$$B(t) = B(0) \quad \exp \quad \int_0^t r(s)ds.$$

A natural interpretation of a riskless asset is that it corresponds to a bank with the (possibly stochastic) **short rate of interest** *r*. An important special case appears when *r* is a deterministic constant, in which case we can interpret *B* as the price of a bond.

We assume that the stock price *S* is given by

$$dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))d\overline{W}(t), \tag{6.2}$$

where $\overline{W}$ is a Wiener process and α and σ are given deterministic functions. The reason for the notation $\overline{W}$, instead of the simpler *W*, will become clear below. The function σ is known as the **volatility** of *S*, while α is the **local mean rate of return** of *S*.

**Remark 6.1.1** Note the difference between the risky stock price *S*, as modeled above, and the riskless asset *B*. The rate of return of *B* is formally given by

$$\frac{dB(t)}{B(t) \cdot dt} = r(t)$$

This object is **locally deterministic** in the sense that, at time *t*, we have complete knowledge of the return by simply observing the prevailing short rate

$r(t)$. Compare this to the rate of return on the stock $S$. This is formally given by

$$\frac{dS(t)}{S(t) \cdot dt} = \alpha(t,S(t)) + \sigma(t,S(t)) \frac{d\bar{W}(t)}{dt},$$

and this is **not** observable at time $t$. It consists of the terms $\alpha\ (t,S(t))$ and $\sigma\ (t,S(t))$, which both are observable at time $t$, plus the "white noise" term $\frac{d\bar{W}(t)}{dt}$, which is random. As opposed to the risk free asset, the stock thus has a **stochastic rate of return** , even on the infinitesimal scale.

The most important special case of the above model occurs when $r$, $\alpha$ and $\sigma$ are deterministic constants. This is the famous **Black–Scholes model** .

**Definition 6.2** *The* **Black–Scholes model** *consists of two assets with dynamics given by*

$$dB(t) = rB(t)dt, \tag{6.3}$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)d\bar{W}(t), \tag{6.4}$$

*where $r$, $\alpha$ and $\sigma$ are deterministic constants.*

# 6.2 Contingent Claims and Arbitrage

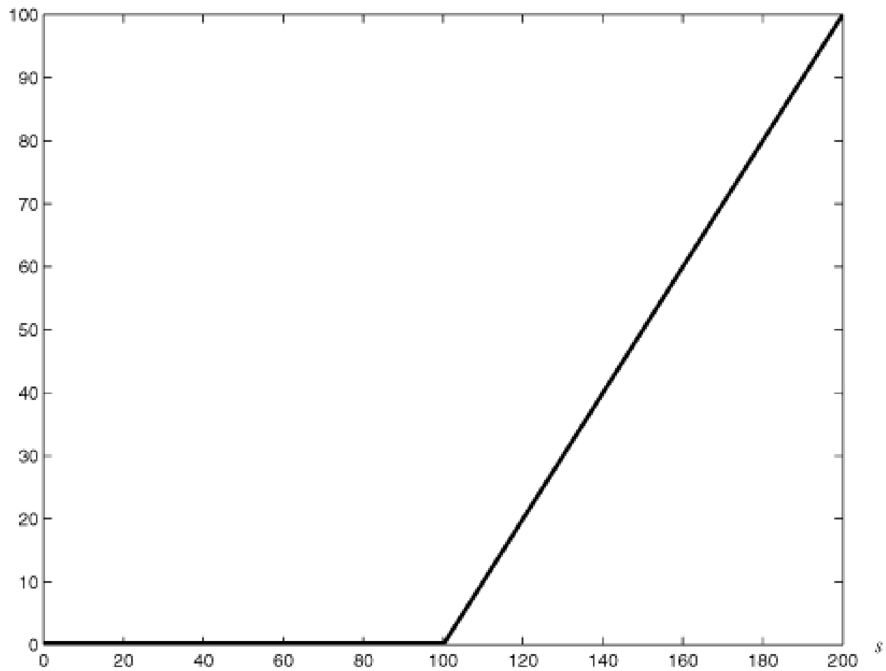
We take as given the model of a financial market given by eqns (6.1)–(6.2), and we now approach the main problem to be studied in this book, namely the pricing of financial derivatives. Later we will give a mathematical definition, but let us at once present the single most important derivative–-the European call option.

**Definition 6.3** *A* **European call option** *with* **exercise price** *(or strike price) K and* **time of maturity** *(exercise date) T on the* **underlying asset** *S is a contract defined by the following clauses.*

- *The holder of the option has, at time T, the right to buy one share of the underlying stock at the price K SEK from the underwriter of the option.*
- *The holder of the option is in no way obliged to buy the underlying stock.*
- *The right to buy the underlying stock at the price K can only be exercised at the precise time T.*

Note that the exercise price $K$ and the time of maturity $T$ are determined at the time when the option is written, which for us typically will be at $t = 0$. A **European put option** is an option which in the same way gives the holder the right to **sell** a share of the underlying asset at a predetermined strike price. For an **American call option** the right to buy a share of the underlying asset can be exercised at any time before the given time of maturity. The common factor of all these contracts is that they all are completely defined in terms of the underlying asset $S$, which makes it natural to call them **derivative instruments** or **contingent claims** . We will now give the formal definition of a contingent claim.

Fig. 6.1 Contract Function. European Call, $K = 100$




**Definition 6.4** *Consider a financial market with vector price process S. A* **contingent claim** *with* **date of maturity** *(exercise date) T, also called a T-claim, is any stochastic variable* $\chi \in \mathcal{F}_T^S$. *A contingent claim* $\chi$ *is called a* **simple** *claim if it is of the form* $\chi = \Phi(S(T))$. *The function* $\Phi$ *is called the* **contract function**.

The interpretation of this definition is that a contingent claim is a contract, which stipulates that the holder of the contract will obtain $\chi$ SEK (which can be positive or negative) at the time of maturity $T$. The requirement that $\chi \in \mathcal{F}_T^S$ simply means that, at time $T$, it will actually be possible to determine the amount of money to be paid out. We see that the European call is a simple contingent claim, for which the contract function is given by

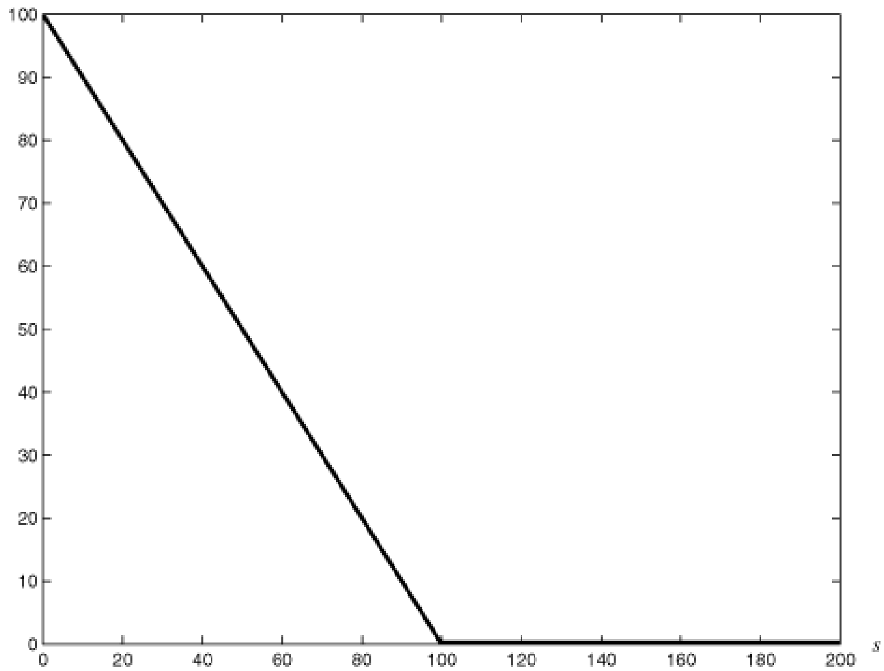
$$\Phi(x) = \max[x - K, 0].$$

The graphs of the contract functions for European calls and puts can be seen in Figs 6.1–6.2. It is obvious that a contingent claim, e.g. like a European call option, is a financial asset which will fetch a price on the market. Exactly how much the option is worth on the market will of course depend on the time $t$ and on the price $S(t)$ of the underlying stock. Our main problem is to determine a “fair” (in some sense) price for the claim, and we will use the standard notation

$$\Pi(t; X) \tag{6.5}$$

for the price process of the claim $\chi$, where we sometime suppress the $\chi$. In the case of a simple claim we will sometimes write $\Pi(t; \Phi)$.

Fig. 6.2 Contract Function. European Put, $K = 100$




If we start at time $T$ the situation is simple. Let us first look at the particular case of a European call.

1. If $S(T) \geq K$ we can make a certain profit by exercising the option in order to buy one share of the underlying stock. This will cost us $K$ SEK. Then we immediately sell the asset on the stock exchange at the price $S(T)$, thus giving us a net profit of $S(T) - K$ SEK.
2. If $S(T) < K$ the option has no value whatsoever.

Thus we see that the only reasonable price $\Pi\ (T)$ for the option at time $T$ is given by

$$\Pi \quad (T) = \max\left[S(T) - K, 0\right]. \tag{6.6}$$

In exactly the same way we see that for a more general contingent claim $\chi$ we have the relation

$$\Pi \quad (T; \chi) = \chi, \tag{6.7}$$

and in the particular case of a simple claim

$$\Pi \quad (T; \chi) = \Phi(S(T)). \tag{6.8}$$

For any time $t < T$ it is, however, far from obvious what the correct price is for a claim $\chi$. On the contrary it seems to be obvious that there is no such thing as a “correct” or “fair” price. The price of an option, like the price of any other asset, is of course determined on the (option) market, and should therefore be an extremely complex aggregate of, for example, the various attitudes to risk on the market and expectations about the future stock prices. It is therefore an extremely surprising fact that, given some fairly mild assumptions, there is a formula (the Black–Scholes formula) which MD>gives the unique price of the option. The main Assumption we will make is that the market is **efficient** in the sense that it is **free of arbitrage possibilities** . We now define this new and central concept.

**Definition 6.5** *An* **arbitrage** *possibility on a financial market is a self-financed portfolio h such that*

$$V^h(0) = 0, \tag{6.9}$$

$$V^h(T) > 0, P - a.s. \tag{6.10}$$

*We say that the market is* **arbitrage free** *if there are no arbitrage possibilities.*

An arbitrage possibility is thus equivalent to the possibility of making a positive amount of money out of nothing with probability 1. It is thus a riskless money making machine or, if you will, a free lunch on the financial market. We interpret an arbitrage possibility as a serious case of mispricing in the market, and our main Assumption is that the market is efficient in the sense that no arbitrage is possible.

**Assumption 6.2.1** *We assume that the price process* $\Pi(t)$ *is such that there are no arbitrage possibilities on the market consisting of* $(B(t), S(t), \Pi(t))$.

A natural question now is how we can identify an arbitrage possibility. The general answer to this question unfortunately requires quite a lot of fairly heavy probabilistic machinery. Happily enough there is a partial result which is sufficient for our purposes.

**Proposition 6.6** *Suppose that there exists a self-financed portfolio h, such that the value process* $V^h$ *has the dynamics*

$$dV^h(t) = k(t)V^h(t)dt, \tag{6.11}$$

*where k is an adapted process. Then it must hold that* $k(t) = r(t)$ *for all t, or there exists an arbitrage possibility.*

**Proof** We sketch the argument, and assume for simplicity that $k$ and $r$ are constant and that $k > r$. Then we can borrow money from the bank at the rate $r$. This money is immediately invested in the portfolio strategy $h$ where it will grow at the rate $k$ with $k > r$. Thus the net investment at $t = 0$ is zero, whereas our wealth at any time $t > 0$ will be positive. In other words we have an arbitrage. If on the other hand $r > k$, we sell the portfolio $h$ short and invest this money in the bank, and again there is an arbitrage. The cases with nonconstant and nondeterministic $r$ and $k$ are handled in the same way.

The main point of the above is that if a portfolio has a value process whose dynamics contain no driving Wiener process, i.e. a **locally riskless porfolio** , then the rate of return of that portfolio must equal the short rate of interest.

To put it in another way, the existence of a portfolio $h$ is for practical purposes equivalent to the existence of a bank with $k$ as its short rate of interest. We can then paraphrase the lemma above by saying that on an arbitrage free market there can only be one short rate of interest.

We now return to the question of how the price process $\Pi\ (t;\chi)$ for a contingent claim $\chi$ can behave, and the main Idea is the following. Since the claim is **defined** entirely in terms of the underlying asset(s), we ought to be able to **price** it in terms of the price of the underlying asset(s) if arbitrage possibilities are to be avoided. Thus we are looking for a way to price the derivative in a way which is **consistent** with the price process of the underlying asset.

To take a simple example, it is quite obvious that for a European call we must have the relation $\Pi\ (t) \leq S(t)$ in an arbitrage free market, because no one in their right mind will buy an option to buy a share at a later date at price $K$ if the share itself can be bought cheaper than the option. For a more formal argument, suppose that at some time $t$ we actually have the relation $\Pi\ (t) > S(t)$. Then we simply sell one option. A part of that money can be used for buying the underlying stock and the rest is invested in the bank (i.e. we buy the riskless asset). Then we sit down and do nothing until time $T$. In this way we have created a self-financed portfolio with zero net investment at time $t$. At time $T$ we will owe max $[S(T) - K,0]$ to the holder of the option, but this money can be paid by selling the stock. Our net wealth at time $T$ will thus be $S(T)-\max\ [S(T) - K,0]$, which is positive, plus the money invested in the bank. Thus we have an arbitrage.

It is thus clear that the requirement of an arbitrage free market will impose some restrictions on the behavior of the price process $\Pi\ (t;\chi)$. This in itself is not terribly surprising. What is surprising is the fact that in the market specified by eqns (6.1)–(6.2) these restrictions are so strong as to completely specify, for any given claim $\chi$, the unique price process $\Pi\ (t;\chi)$ which is consistent with absence of arbitrage. For the case of **simple** contingent claims the formal argument will be given in the next section, but we will now give the general idea.

To start with, it seems reasonable to assume that the price $\Pi\ (t;\chi)$ at time $t$ in some way is determined by expectations about the future stock price $S(T)$. Since $S$ is a Markov process such expectations are in their turn based on the present value of the price process (rather than on the entire trajectory on $[0,t]$). We thus make the following assumption.

**Assumption 6.2.2** *We assume that*

1. *The derivative instrument in question can be bought and sold on a market.*
2. *The market is free of arbitrage.*
3. *The price process for the derivative asset is of the form*

$$\Pi\quad(t;\chi) = F(t,S(t)), \tag{6.12}$$

*where $F$ is some smooth function.*

Our task is to determine what $F$ might look like if the market consisting of $S(t)$, $B(t)$ and $\Pi\ (t;\ \chi)$ is arbitrage free. Schematically we will proceed in the following manner.

1. Consider $\alpha$, $\sigma$, $\Phi$, $F$ and $r$ as exogenously given.
2. Use the general results from Section 5.2 to describe the dynamics of the value of a hypothetical self-financed portfolio based on the derivative instrument and the underlying stock (nothing will actually be invested in or loaned by the bank).
3. It turns out that, by a clever choice, we can form a self-financed portfolio whose value process has a stochastic differential without any driving Wiener process. It will thus be of the form (6.11) above.
4. Since we have assumed absence of arbitrage we must have $k = r$.
5. The Condition $k = r$ will in fact have the form of a partial differential equation with $F$ as the unknown function. In order for the market to be efficient $F$ must thus solve this PDE.
6. The equation has a unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

# 6.3 The Black–Scholes Equation

In this section we will carry through the schematic argument given in the previous section. We assume that the a priori given market consists of two assets with dynamics given by

$$dB(t) = rB(t)dt, \tag{6.13}$$

$$dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))d\bar{W}(t), \tag{6.14}$$

where the short rate of interest $r$ is a deterministic constant. We consider a simple contingent claim of the form

$$\chi = \Phi(S(T)), \tag{6.15}$$

and we assume that this claim can be traded on a market and that its price process $\Pi\ (t) = \Pi\ (t;\ \Phi)$ has the form

$$\Pi\ (t) = F(t,S(t)), \tag{6.16}$$

for some smooth function $F$. Our problem is to find out what $F$ must look like in order for the market $[S(t), B(t), \Pi\ (t)]$ to be free of arbitrage possibilities.

We start by computing the price dynamics of the derivative asset, and the Itô formula applied to (6.16) and (6.14) gives us

$$d\Pi\ (t) = \alpha_\pi(t)\Pi\ (t)dt + \sigma_\pi(t)\Pi\ (t)d\bar{W}(t), \tag{6.17}$$

where the processes $\alpha_\pi(t)$ and $\sigma_\pi(t)$ are defined by

$$\alpha_\pi(t) = \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}, \tag{6.18}$$

$$\sigma_\pi(t) = \frac{\sigma S F_s}{F}.$$

Here subscripts denote partial derivatives, and we have used a shorthand notation of the form

(6.19)

$$\frac{\sigma S F_s}{F} = \frac{\sigma(t,S(t))S(t)F_s(t,S(t))}{F(t,S(t))},$$

and similarly for the other terms above.

Let us now form a portfolio based on two assets: the underlying stock and the derivative asset. Denoting the relative portfolio by $(u_s,u_\pi)$ and using eqn (5.13) we obtain the following dynamics for the value $V$ of the portfolio.

$$dV = V\left\{u_s\left[\alpha dt + \sigma d\bar{W}\right] + u_\pi\left[\alpha_\pi dt + \sigma_\pi d\bar{W}\right]\right\}$$

where we have suppressed $t$. We now collect $dt$- and $d\bar{W}$-terms to obtain

(6.20)

$$dV = V\left[u_s\alpha + u_\pi\alpha_\pi\right]dt + V\left[u_s\sigma + u_\pi\sigma_\pi\right]d\bar{W}.$$

(6.21)

The point to notice here is that both brackets above are linear in the arguments $u_s$ and $u_\pi$. Recall furthermore that the only restriction on the relative portfolio is that we must have

$$u_s + u_\pi = 1,$$

for all $t$. Let us thus define the relative portfolio by the linear system of equations

$$u_s + u_\pi = 1,$$

$$u_s\sigma + u_\pi\sigma_\pi = 0. \qquad (6.22)$$

(6.23)

Using this portfolio we see that by its very definition the driving $dW$-term in the $V$-dynamics of eqn (6.21) vanishes completely, leaving us with the equation

$$dV = V\left[u_s\alpha + u_\pi\alpha_\pi\right]dt.$$

(6.24)

Thus we have obtained a locally riskless portfolio, and because of the requirement that the market is free of arbitrage, we may now use Proposition 6.6 to deduce that we must have the relation

$$u_s\alpha + u_\pi\alpha_\pi = r.$$

This is thus the Condition for absence of arbitrage, and we will now look more closely at this equation.

(6.25)

It is easily seen that the system (6.22)–(6.23) has the solution

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma},$$

(6.26)

$$u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma},$$

which, using (6.19), gives us the portfolio more explicitly as

(6.27)

$$u_S(t) = \frac{S(t)F_s(t,S(t))}{S(t)F_s(t,S(t)) - F(t,S(t))},$$

(6.28)

$$u_\pi(t) = \frac{-F(t,S(t))}{S(t)F_s(t,S(t)) - F(t,S(t))}.$$

(6.29)

Now we substitute (6.18), (6.28) and (6.29) into the absence of arbitrage Condition (6.25). Then, after some calculations, we obtain the equation

$$F_t(t,S(t)) + rS(t)F_s(t,S(t)) + \frac{1}{2}\sigma^2(t,S(t))S^2(t)F_{ss}(t,S(t) - rF(t,S(t)) = 0.$$

Furthermore, from the previous section we must have the relation

$$\Pi \quad (T) = \Phi(S(T)).$$

These two equations have to hold with probability 1 for each fixed $t$. Furthermore it can be shown that under very weak assumptions (which trivially are satisfied in the Black–Scholes model) the distribution of $S(t)$ for every fixed $t > 0$ has support on the entire positive real line. Thus $S(t)$ can take any value whatsoever, so $F$ has to satisfy the following (deterministic) PDE.

$$\begin{aligned} F_t(t,s) + rsF_s(t,s) + \frac{1}{2}s^2\sigma^2(t,s)F_{ss}(t,s) - rF(t,s) &= 0, \\ F(T,s) &= \Phi(s). \end{aligned}$$

Summing up these results we have proved the following proposition, which is in fact the most central result in the book.

**Theorem 6.7 (Black–Scholes equation)** *Assume that the market is specified by eqns (6.13)–(6.14) and that we want to price a contingent claim of the form (6.15). Then the only pricing function of the form (6.16) which is consistent with the absence of arbitrage is when F is the solution of the following boundary value problem in the domain* $[0,T] \times R_+$.

$$F_t(t,s) + rsF_s(t,s) + \frac{1}{2}s^2\sigma^2(t,s)F_{ss}(t,s) - rF(t,s) = 0,$$

$$F(T,s) = \Phi(s). \quad (6.30)$$

(6.31)

Before we go on to a closer study of the pricing equation (6.30) let us make a few comments.

Firstly it is important to stress the fact that we have obtained the price of the claim $\chi$ in the form $\Pi\ (t; \chi) = F(t,S(t))$, i.e. the price of the claim is given as a function of the price of the underlying asset $S$. This is completely in line with the basic Idea explained earlier, that the pricing of derivative assets is a question of pricing the derivative in a way which is

**consistent** with the price of the underlying asset. We are thus **not** presenting an absolute pricing formula for $\chi$. On the contrary, derivative pricing is all about **relative** pricing, i.e. pricing the derivative asset **in terms of** the price of the underlying asset. In particular this means that in order to use the technique of arbitrage pricing at all we must have one or several underlying price processes given a priori.

Secondly a word of criticism. At a first glance our derivation of the pricing equation (6.30) seems to be fairly convincing, but in fact it contains some rather weak points. The logic of the argument was that we **assumed** that the price of the derivative was a function $F$ of $t$ and $S(t)$. Using this Assumption we then showed that in an arbitrage free market $F$ had to satisfy the Black–Scholes equation. The question now is if we really have good reasons to assume that the price is of the form $F(t,S(t))$. The Markovian argument given above sounds good, but it is not totally convincing.

A much more serious objection is that we assume that there actually exists a market for the derivative asset, and in particular that there exists a price process for the derivative. This Assumption of an existing market for the derivative is crucial for the argument since we are actually constructing a portfolio based on the derivative (and the underlying asset). If the derivative is not traded then the portfolio cannot be formed and our argument breaks down. The Assumption of an existing price for the derivative is of course innocent enough in the case of a standard derivative, like a European call option, which *de facto* is traded in large volumes. If, however, we want to price an OTC ("over the counter") instrument, i.e. an instrument which is not traded on a regular basis, then we seem to be in big trouble.

Happily enough there is an alternative argument for the derivation of the pricing equation (6.30), and this argument (which will be given below) is not open to the criticism above. The bottom line is that the reader can feel safe: equation (6.30) really is the "correct" equation.

Let us end by noting an extremely surprising fact about the pricing equation, namely that it does not contain the local mean rate of return $\alpha\ (t,s)$ of the underlying asset. In particular this means that, when it comes to pricing derivatives, the local rate of return of the underlying asset plays no role whatsoever. The only aspect of the underlying price process which is of any importance is the volatility $\sigma\ (t,s)$. Thus, for a given volatility, the price of a fixed derivative (like a European call option) will be exactly the same regardless of whether the underlying stock has a 10%, a 50%, or even a −50% rate of return. At a first glance this sounds highly counter-intuitive and one is tempted to doubt the whole procedure of arbitrage pricing. There is, however, a natural explanation for this phenomenon, and we will come back to it later. At this point we can only say that the phenomenon is closely connected to the fact that we are pricing the derivative in terms of the price of the underlying asset.

## 6.4 Risk Neutral Valuation

Let us again consider a market given by the equations

$$dB(t) = rB(t)dt, \tag{6.32}$$

$$dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))d\bar{W}(t), \tag{6.33}$$

and a contingent claim of the form $\chi = \Phi\ (S(T))$. Then we know that the arbitrage free price is given by $\Pi\ (t;\ \Phi) = F(t,S(t))$ where the function $F$ is the solution of the pricing equation (6.30)–(6.31). We now turn to the question of actually solving the pricing equation and we notice that this equation is precisely of the form which can be solved using a stochastic representation formula à la Feynman–Kač. Using the results from Section 4.5 we see that the solution is given by

$$F(t,s) = e^{-r(T-t)} E_{t,s}[\Phi(X(T))], \tag{6.34}$$

where the $X$ process is defined by the dynamics

$$dX(u) = rX(u)du + X(u)\sigma(u,X(u))dW(u), \tag{6.35}$$

$$X(t) = s, \tag{6.36}$$

where $W$ is a Wiener process. The important point to note here is that the SDE (6.35) is of precisely the same form as that of the price process $S$. The only, but important, change is that whereas $S$ has the local rate of return $\alpha$, the $X$-process has the short rate of interest $r$ as its local rate of return.

The $X$-process above is logically just a technical tool, defined for the moment, and in particular we can name it as we please. In view of the resemblance between $X$ and $S$ it is rather tempting to call it $S$ instead of $X$. This is perfectly acceptable as long as we do not confuse the "real" $S$-process of (6.33) with the "new" $S$-process, and one way to achieve this goal is by the following procedure.

Let us agree to denote the "objective" probability measure which governs our real model (6.32)–(6.33) by the letter $P$. Thus we say that the $P$-dynamics of the $S$-process are that of (6.33). We now define another probability measure $Q$ under which the $S$-process has a different probability distribution. This is done by defining the $Q$-dynamics of $S$ as

$$dS(t) = rS(t)dt + S(t)\sigma(t,S(t))dW(t), \tag{6.37}$$

where $W$ is a $Q$-Wiener process. In order to distinguish the measure under which we take expectations we introduce some notational conventions.

**Notational convention 6.4.1** *For the rest of the text, the following conventions will be used.*

- *We identify the expectation operator by letting $E$ denote expectations taken under the $P$-measure whereas $E^Q$ denotes expectations taken under the $Q$-measure.*
- *We identify the Wiener process. Thus $\bar{W}$ will denote a $P$-Wiener process, whereas $W$ will denote a $Q$-Wiener process.*

The convention on $W$ has the advantage that it is possible, at a glance, to decide under which measure a certain SDE is given. We will work much more often under $Q$ than under $P$, and this is the reason why the $Q$-Wiener process $W$ has a simpler notation than the $P$-Wiener process $\bar{W}$. Using this notation we may now state the following central result for derivative pricing.

**Theorem 6.8** *The arbitrage free price of the claim* $\Phi(S(T))$ *is given by* $\Pi(t;\Phi) = F(t,S(t))$, *where* $F$ *is given by the formula*

$$F(t,s) = e^{-r(T-t)} E^{Q}_{t,s}\left[\Phi(S(T))\right], \tag{6.38}$$

*where the* $Q$*-dynamics of* $S$ *are those of* (*6.37*).

There is a natural economic interpretation of the formula (6.38). We see that the price of the derivative, given today's date $t$ and today's stock price $s$, is computed by taking the expectation of the final payment $E^{Q}_{t,s}[\Phi(S(T))]$ and then discounting this expected value to present value using the discount factor $e^{-r(T-t)}$. The important point to note is that when we take the expected value we are **not** to do this using the objective probability measure $P$. Instead we shall use the $Q$-measure defined in (6.37). This $Q$-measure is sometimes called the **risk adjusted measure** but most often it is called the **martingale measure** , and this will be our terminology. The reason for the name is that under $Q$ the discounted process

$$\frac{S(t)}{B(t)}$$

turns out to be a $Q$-martingale. In a deeper investigation of arbitrage pricing the $Q$-measure is the fundamental object of study. We formulate the martingale property as a separate result.

**Proposition 6.9 (Risk neutral valuation)** *In the Black–Scholes model, the price process* $\Pi(t)$ *for every traded asset, be it the underlying or derivative asset, has the property that the normalized price process*

$$Z(t) = \frac{\Pi(t)}{B(t)}$$

*is a martingale under the measure* $Q$.

**Proof** See the exercises.

The formula (6.38) is sometimes referred to as the formula of **risk neutral valuation** . Suppose that all agents are risk neutral. Then all assets will command a rate of return equal to the short rate of interest, i.e. in a risk neutral world the stock price will actually have the $Q$-dynamics above (more precisely, in this case we will have $Q = P$). Furthermore, in a risk neutral world the present value of a future stochastic payout will equal the expected value of the net payments discounted to present value using the short rate of interest. Thus formula (6.38) is precisely the kind of formula which would be used for valuing a contingent claim in a risk neutral world. Observe, however, that we do **not** assume that the agents in our model are risk neutral. The formula only says that the value of the contingent claim can be calculated **as if** we live in a risk neutral world. In particular the agents are allowed to have any attitude to risk whatsoever, as long as they all prefer a larger amount of (certain) money to a lesser amount. Thus the valuation formula above is **preference free** in the sense that it is valid regardless of the specific form of the agents' preferences.

# 6.5 The Black–Scholes Formula

In this section we specialize the model of the previous section to the case of the Black–Scholes model,

$$dB(t) = rB(t)dt,$$

$$dS(t) = \alpha S(t)dt + \sigma S(t)d\overline{W}(t), \tag{6.39}$$

(6.40)

where α and σ are constants. From the results of the previous section we know that the arbitrage free price of a simple claim Φ ($S(T)$) is given by

$$F(t,s) = e^{-r(T-t)}E^{Q}_{t,s}[\Phi(S(T))],$$

where the $Q$-dynamics of $S$ are given by

(6.41)

$$dS(u) = rS(u)du + \sigma S(u)dW(u),$$

$$S(t) = s. \tag{6.42}$$

(6.43)

In this SDE we recognize our old friend geometric Brownian motion from Section 4.2. Using the results from Section 4.2 we can thus write $S(T)$ explicitly as

$$S(T) = s \quad \exp \quad \left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(W(T) - W(t))\right\}.$$

Thus we can write $S(T) = se^Y$ where $Y$ is a stochastic variable with the distribution

(6.44)

$$N\left[\left(r - \frac{1}{2}\sigma^2\right)(T - t), \sigma\sqrt{T - t}\right],$$

and we obtain the following pricing formula

$$F(t,s) = e^{-r(T-t)}\int_{-\infty}^{\infty} \Phi \quad (se^y)f(y)dy,$$

(6.45)

where $f$ is the density function for the stochastic variable $Y$ above.

Formula (6.45) is an integral formula which, for a general choice of contract function Φ, must be evaluated numerically. There are, however, a few particular cases where we can evaluate (6.45) more or less analytically, and the best known of these is the case of a European call option, where Φ has the form Φ ($x$) = max [$x - K$,0]. In this case it is convenient to normalize, and it is easy to see that we can write $S(T)$ as

$$S(T) = se^{\tilde{r}\tau + \sigma\sqrt{\tau}Z},$$

where $\tilde{r} = (r - \frac{1}{2}\sigma^2), \quad \tau = T - t$ and $Z$ is a standardized normal variable. The integral in (6.45) now becomes

$$\int_{-\infty}^{\infty} \max\left[se^{\tilde{r}\tau + \sigma\sqrt{\tau}z} - K, 0\right]\varphi(z)dz$$

where $\varphi$ is the density of the $N[0,1]$ distribution, i.e.

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

The integrand in the integral above vanishes when

$$se^{\tilde{r}\tau + \sigma\sqrt{\tau}z} < K,$$

i.e. when $z < z_0$, where

$$z_0 = \frac{\ln\left(\frac{K}{s}\right) - \tilde{r}\tau}{\sigma\sqrt{\tau}}.$$

The integral can thus be written

$$\begin{aligned}\int_{z_0}^{\infty}\left(se^{\tilde{r}\tau + \sigma\sqrt{\tau}z} - K\right)\varphi(z)dz &= \int_{z_0}^{\infty} se^{\tilde{r}\tau + \sigma\sqrt{\tau}z}\varphi(z)dz - \int_{z_0}^{\infty} K\varphi(z)dz \\ &= A - B\end{aligned}$$

The integral $B$ can obviously (why?) be written as

$$K \cdot \text{Prob} \quad (Z \geq z_0),$$

and, using the symmetry of the $N[0,1]$ distribution, this can be written as

$$K \cdot \text{Prob} \quad (Z \leq -z_0).$$

We denote, as is common, the cumulative distribution function of the $N[0,1]$ distribution by $N$, i.e.

$$N[x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz.$$

Thus we can write $B$ as

$$B = K \cdot N[-z_0].$$

In the integral $A$ we complete the square in the exponent to obtain

$$\begin{aligned}A &= \frac{se^{\tilde{r}\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{\sigma\sqrt{\tau}z - \frac{1}{2}z^2} dz = \frac{se^{\tilde{r}\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{\tau})^2 + \frac{1}{2}\sigma^2\tau} dz \\ &= \frac{se^{r\tau}}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{\tau})^2} dz.\end{aligned}$$

Here we recognize the density of a $N[\sigma\sqrt{\tau}, 1]$ distribution, so we have

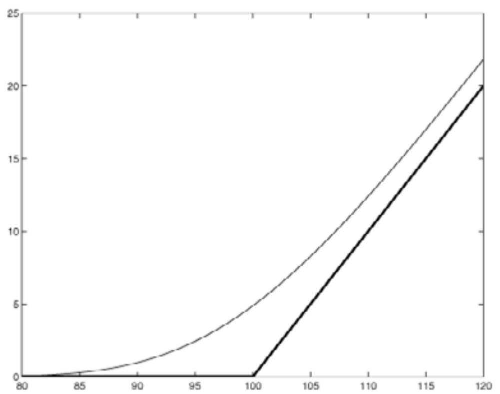
$$A = se^{r\tau} \cdot \text{Prob} \quad (Z' \geq z_0)$$

where $Z$ is $N[\sigma\sqrt{\tau},1]$ . Normalizing $Z$ to a $N$[0,1] variable, and again using symmetry, we can write

$$A = se^{r\tau} \cdot N[-z_0 + \sigma\sqrt{\tau}].$$

We have thus computed the integral in (6.45), and we finally have the following famous result, which is known as the **Black–Scholes formula** .

Fig. 6.3 The Black–Scholes Price Of a Call Option: $K = 100$, $\sigma = 0.2$, $T - t = 0.25$




**Proposition 6.10 (Black–Scholes formula)** *The price of a European call option with strike price K and time of maturity T is given by the formula* $\Pi(t) = F(t,S(t))$, *where*

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)].$$

*Here N is the cumulative distribution function for the* $N$ [0,1] *distribution and*

(6.46)

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\},$$

(6.47)

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}.$$

(6.48)

Figure 6.3 shows the graph of the Black–Scholes pricing function (the unit of time is chosen to be one year).

# 6.6 Options on Futures

The purpose of this section is to derive the Black formulas for options written on a futures contract. Our discussion here will be rather brief, and for more institutional and technical information the reader is referred to Chapter 20 (and the Notes), where the contracts are discussed in more detail.

## 6.6.1 Forward Contracts

Consider a standard Black–Scholes model, a simple $T$-claim $\chi = \Phi(S_T)$, and assume that we are standing at time $t$. A **forward contract** on $\chi$, made at $t$, is a contract which stipulates that the holder of the contract pays the deterministic amount $K$ at the **delivery date** $T$, and receives the stochastic amount $\chi$ at $T$. Nothing is paid or received at the time $t$, when the contract is made. Note that **forward price** $K$ is determined already at time $t$. It is customary to use the notation $K = f(t;T,\chi)$, and our primary concern is to compute $f(t;T,\chi)$.

This is, however, easily done. We see that the entire forward contract is a contingent $T$-claim $Y$ of the form

$$Y = \chi - K,$$

and, by definition, the value of $Y$ at the time $t$ when the contract is made equals zero. Thus we have

$$\Pi \quad (t; \chi - K) = 0,$$

which leads to

$$\Pi \quad (t; \chi) = \Pi \quad (t; K).$$

Using risk neutral valuation we immediately have $\Pi\ (t;K) = e^{-r(T-t)}K$ and $\Pi \quad (t; \chi) = e^{-r(T-t)} \times E^{Q}_{t,s}[\chi]$, so we have proved the first part of the following result. The second part is left as an exercise.

**Proposition 6.11** *The forward price $f(t;T, \chi)$, contracted at t, on the T-claim $\chi$ is given by*

$$f(t; T, \chi) = E^{Q}_{t,s}[\chi]. \tag{6.49}$$

*In particular, if $\chi = S_T$ the corresponding forward price, denoted by $f(t;T)$, is given by*

$$f(t; T) = e^{r(T-t)}S_t. \tag{6.50}$$

**Remark 6.6.1** Note the difference between the forward price $f(t;T, \chi)$ which is a sum to be paid at $T$, for a forward contract entered at time $t$, and the spot price of the entire forward contract. This latter price is zero at the time $t$ when the contract is made, but at any subsequent time $s > t$ it will typically have a nonzero value.

## 6.6.2 Futures Contracts and the Black Formula

With the same setup as the previous section we will now discuss a **futures contract** on $\chi$. This contract is very close to the corresponding forward contract in the sense that it is still a contract for the delivery of $\chi$ at $T$. The difference is that all the payments, from the holder of the contract to the underwriter, are no longer made at $T$. Let us denote the futures price by $F(t;T,\chi)$; the payments are delivered continuously over time, such that the holder of the contract over the time interval $[s,s + \Delta s]$ receives the amount

$$F(s + \Delta s; T,\chi) - F(s; T,\chi)$$

from the underwriter. Finally the holder will receive $\chi$, and pay $F(T;T,\chi)$, at the **delivery date** $T$. By definition, the (spot) price (at any time) of the entire futures contract equals zero. Thus the cost of entering or leaving a futures contract is zero, and the only contractual obligation is the payment stream described above. See Chapter 20 for more details, and for a proof of the following result.

**Proposition 6.12** *If the short rate is deterministic, then the forward and the futures price processes coincide, and we have*

$$F(t;T,\chi)=E_{t,s}^{Q}[\chi]. \tag{6.51}$$

We will now study the problem of pricing a European call option, with exercise date $T$, and exercise price $K$, on an underlying futures contract. The futures contract is a future on $S$ with delivery date $T_1$, with $T < T_1$. Options of this kind are traded frequently, and by definition the holder of this option will, at the exercise time $T$, obtain a long position in the futures contract, plus the stochastic amount

$$\chi=\max\left[F(T;T_1)-K,0\right]. \tag{6.52}$$

Since the spot price of the futures contact equals zero, we may, for pricing purposes, forget about the long futures position embedded in the option, and identify the option with the claim in (6.52).

We now go on to price the futures option, and we start by using Proposition 6.12 and eqn (6.50) in order to write

$$\chi=e^{r(T_1-T)}\max\left[S_T-e^{-r(T_1-T)}K,0\right].$$

Thus we see that the futures option consists of $e^{r(T_1-T)}$ call options on the underlying asset $S$, with exercise date $t$ and exercise price $e^{-r(T_1-T)}K$. Denoting the price at $T$ of the futures option by $c$, the stock price at $t$ by $s$, and the futures price $F(t;T_1)$ by $F$, we thus have, from the Black–Scholes formula,

$$c=e^{r(T_1-T)}\left[sN[d_1]-e^{-r(T-t)}e^{-r(T_1-T)}KN[d_2]\right]$$

where $d_1$ and $d_2$ are obtained from the Black–Scholes $d_1$ and $d_2$ by replacing $k$ with $e^{-r(T_1-T)}K$. Finally we may substitute $s=Fe^{-r(T_1-t)}$, and simplify, to obtain the so called "Black-76 formula".

**Proposition 6.13 (Black's formula)** *The price, at t, of a European call option, with exercise date T and exercise price K, on a futures contract (on an underlying asset price S) with delivery date $T_1$ is given by*

$$c=e^{-r(T-t)}\left[FN[d_1]-KN[d_2]\right],$$

*where F is the futures price $F = F(T,T_1)$, and*

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1-\sigma\sqrt{T-t}. \end{aligned} \tag{6.53}$$

# 6.7 Volatility

In order to be able to use the theory derived above in a concrete situation, we need to have numerical estimates of all the input parameters. In the Black–Scholes model the input data consists of the string $s$, $r$, $T$, $t$ and $\sigma$. Out of these five parameters, $s$, $r$, $T$ and $t$ can be observed directly, which leaves us with the problem of obtaining an estimate of the volatility σ. Here there are two basic approaches, namely to use "historic volatility" or "implied volatility".

## 6.7.1 Historic Volatility

Suppose that we want to value a European call with six months left to maturity. An obvious Idea is to use historical stock price data in order to estimate σ. Since, in real life, the volatility is not constant over time, one standard practice is to use historical data for a period of the same length as the time to maturity, which in our case means that we use data for the last six months.

In order to obtain an estimate of σ we assume that we have the standard Black–Scholes GBM model (6.4) under the objective measure $P$. We sample (observe) the stock price process $S$ at $n+1$ discrete equidistant points $t_0, t_1, \ldots, t_n$, where $\Delta t$ denotes the length of the sampling interval, i.e. $\Delta t = t_i - t_{i-1}$.

We thus observe $S(t_0), \ldots, S(t_n)$, and in order to estimate σ we use the fact that $S$ has a log-normal distribution. Let us therefore define $\xi_1, \ldots, \xi_n$ by

$$\xi_i = \ln \left( \frac{S(t_i)}{S(t_{i-1})} \right).$$

From (4.15) we see that $\xi_1, \ldots, \xi_n$ are independent, normally distributed random variables with

$$E[\xi_i] = \left( \alpha - \frac{1}{2}\sigma^2 \right)\Delta t,$$

$$Var[\xi_i] = \sigma^2 \Delta t.$$

Using elementary statistical theory we see that an estimate of σ is given by

$$\sigma^* = \frac{S_\xi}{\sqrt{\Delta t}},$$

where the sample variance $S_\xi^2$ is given by

$$S_\xi^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( \xi_i - \bar{\xi} \right)^2,$$

$$\bar{\xi} = \frac{1}{n} \sum_{i=1}^{n} \xi_i.$$

The standard deviation, $D$, of the estimate $\sigma^*$ is approximatively given by

$$D(\sigma^*) \approx \frac{\sigma^*}{\sqrt{2n}}.$$

### 6.7.2 Implied Volatility

Suppose again that we want to value a European call with six months left to maturity. An argument against the use of historical volatility is that in real life volatility is not constant, but changes over time, and thus we want an estimate of the volatility for the coming six months. Using historical volatility we will, however, only obtain an estimate for the volatility over the past six months. If, furthermore, our objective is to price our option **consistently** with respect to other assets which are already priced by the market, then we really should use the **market expectation** of the volatility for the next six months.

One way of finding the market expectation of the volatility is by getting market price data for another six month "benchmark" option, written on the same underlying stock as the option which we want to value. Denoting the price of the benchmark option by $p$, the strike price by $K$, today's observed value of the underlying stock by $s$, and writing the Black–Scholes pricing formula for European calls by $c(s,t,T,r,\sigma, K)$, we then solve the following equation for $\sigma$

$$p = c(s,t,T,r,\sigma,K).$$

In other words, we try to find the value of $\sigma$ which the market has implicitly used for valuing the benchmark option. This value of $\sigma$ is called the **implied volatility** , and we then use the implied volatility for the benchmark in order to price our original option. Put another way, we price the original option in terms of the benchmark.

We note that implied volatilities can be used to test (in a nonstandard way) the Black–Scholes model. Suppose, for example, that we observe the market prices of a number of European calls with the same exercise date on a single underlying stock. If the model is correct (with a constant volatility) then, if we plot implied volatility as a function of the exercise price, we should obtain a straight line. Contrary to this, it is often empirically observed that options far out of the money (see Remark below) or deep into the money are traded at higher implied volatilities than options at the money. The graph of the observed implied volatility function thus often looks like the smile of the Cheshire cat, and for this reason the implied volatility curve is termed the **volatility smile** .

**Remark 6.7.1** A call option is said to be "in the money" at time $t$ if $S_t > K$, and "out of the money" if $S_t < K$. For put options the inequalities are reversed. If $S_t = K$ the option is said to be "at the money".

## 6.8American Options

Up to now we have assumed that a contract, like a call option, can only be exercised exactly at the exercise time $T$. In real life a large number of options can in fact be exercised at **any time prior to $T$**. The choice of exercise time is thus left to the holder of the contract, and a contract with this feature is called an **American** contract.

To put it more formally, let us fix a final exercise date $T$ and a contract function $\Phi$. The European version of this contract will, as usual, pay the amount $\Phi(S_T)$ at time $T$ to the holder of the contract. If the contract, on the other hand, is of the American type, then the holder will obtain the amount $\Phi(S_t)$ if he/she chooses to exercise the contract at time $t$. The situation is complicated further by the fact that the exercise time $t$ does not have to be chosen a priori (i.e. at $t = 0$). It can be chosen on the basis of the information generated by the stock price process, and thus the holder will in fact choose a **random exercise time** $\tau$. The exercise time (or rather exercise strategy) $\tau$ has to be chosen such that the decision on whether to exercise the contract at time $t$ or not, depends only upon the information generated by the price process up to time $t$. The mathematical formulation of this property is in terms of so called "stopping times", but we will not go further into this subject.

American contracts are thus more complicated to analyze than their European counterparts, since the holder of the contract has to decide on an **optimal exercise strategy** . Mathematically this means that we have to solve the "optimal stopping problem"

$$\max_{\tau} \left[ E^Q\left[ e^{-r\tau} \Phi(S_\tau) \right] \right],$$

where $\tau$ is allowed to vary over the class of stopping times. Problems of this kind are quite hard to solve, and analytically they lead to so called "free boundary value problems" (or variational inequalities) instead of the corresponding parabolic PDEs for the European counterparts. The mathematics of this lies outside the scope of this book, but it may be of interest to know that for American contracts practically no analytical formulas are at hand. See the Notes below for references.

One situation, however, is very easy to analyze, even for American contracts, and that is the case of an American call option on a nondividend paying underlying stock. Let us consider an American call option with final exercise date $T$ and exerise price $K$. We denote the pricing function for the American option by $C(t,s)$ and the pricing function for the corresponding European option (with the same $T$ and $K$) by $c(t,s)$.

Firstly we note that we have (why?) the trivial inequality

$$C(t,s) \geq c(t,s). \tag{6.54}$$

Secondly we, have for all $t < T$, the less obvious inequality.

$$c(t,s) \geq s - Ke^{-r(T-t)}. \tag{6.55}$$

To see why this inequality holds it is sufficient to consider two portfolios, $A$ and $B$. $A$ consists of a long position in the European option, whereas $B$ consists of a long position in the underlying stock and a loan expiring at $T$, with face value $K$. Denoting the price of $A$ and $B$ at any time $t$ by $A_t$ and $B_t$ respectively, it is easily seen that $A_T \geq B_T$ regardless of the value of $S_T$ (analyze the two cases

$S_T \geq K$ and $S_T < K$). In order to avoid arbitrage possibilities we then must have $A_t \geq B_t$ for all $t \leq T$, which is precisely the content of (6.55).

Furthermore, assuming a positive rate of interest, we have the trivial inequality

$$s - Ke^{-r(T-t)} > s - K, \quad \forall t < T,$$

so we end up with the inequality

$$C(t,s) > s - K, \quad \forall t < T. \tag{6.56}$$

On the left hand side we have the value of the American option at time $t$, whereas the right hand side gives us the value of actually exercising the option at time $t$. Since the value of the option is strictly greater than the value of exercising the option, it can thus not be optimal to exercise the option at time $t$. Since this holds for all $t < T$, we see that it is in fact never optimal to exercise the option before $T$, and we have the following result.

**Proposition 6.14** *Assume that* $r > 0$. *For an American call option, written on an underlying stock without dividends, the optimal exercise time* $\tau$ *is given by* $\tau = T$. *Thus the price of the American option coincides with the price of the corresponding European option.*

For American call options with discrete dividends, the argument above can be extended to show that it can only be optimal to exercise the option either at the final time $T$ or at one of the dividend times. The American put option (even without dividends) presents a hard problem without an analytical solution. See the Notes below.

## 6.9 Exercises

**Exercise 6.1** Consider the standard Black–Scholes model and a $T$-claim $\chi$ of the form $\chi = \Phi(S(T))$. Denote the corresponding arbitrage free price process by $\Pi(t)$.

(a) Show that, under the martingale measure $Q$, $\Pi(t)$ has a local rate of return equal to the short rate of interest $r$. In other words show that $\Pi(t)$ has a differential of the form

$$d\Pi(t) = r \cdot \Pi(t)dt + g(t)dW(t).$$

**Hint:** Use the $Q$-dynamics of $S$ together with the fact that $F$ satisfies the pricing PDE.

(b) Show that, under the martingale measure $Q$, the process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a **martingale** . More precisely, show that the stochastic differential for $Z$ has zero drift term, i.e. it is of the form

$$dZ(t) = Z(t)\sigma_Z(t)dW(t).$$

Determine also the diffusion process $\sigma_Z(t)$ (in terms of the pricing function $F$ and its derivatives).

**Exercise 6.2** Consider the standard Black–Scholes model. An innovative company, *F& H INC*, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the *GL*. The holder of a *GL* with maturity time *T*, denoted as *GL*(*T*), will, at time *T*, obtain the sum ln $S(T)$. Note that if $S(T) < 1$ this means that the holder has to pay a positive amount to *F& H INC*. Determine the arbitrage free price process for the *GL*(*T*).

**Exercise 6.3** Consider the standard Black–Scholes model. Derive the Black–Scholes formula for the European call option.

**Exercise 6.4** Consider the standard Black–Scholes model. Derive the arbitrage free price process for the *T*-claim $\chi$ where $\chi$ is given by $\chi = \{S(T)\}^{\beta}$. Here $\beta$ is a known constant.

**Hint:** For this problem you may find Exercises 4.5 and 3.4 useful.

**Exercise 6.5** A so called **binary option** is a claim which pays a certain amount if the stock price at a certain date falls within some prespecified interval. Otherwise nothing will be paid out. Consider a binary option which pays *K* SEK to the holder at date *T* if the stock price at time *T* is in the inerval $[\alpha, \beta]$. Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function *N*.

**Exercise 6.6** Consider the standard Black–Scholes model. Derive the arbitrage free price process for the claim $\chi$ where $\chi$ is given by $\chi = \frac{S(T_1)}{S(T_0)}$. The times $T_0$ and $T_1$ are given and the claim is paid out at time $T_1$.

**Exercise 6.7** Consider the American corporation *ACMEINC*. The price process *S* for *ACME* is of course denoted in US$ and has the *P*-dynamics

$$dS = \alpha S dt + \sigma S d\bar{W}_1,$$

where $\alpha$ and $\sigma$ are known constants. The currency ratio SEK/US$ is denoted by *Y* and *Y* has the dynamics

$$dY = \beta Y \quad dt + \delta Y \quad d\bar{W}_2,$$

where $\bar{W}_2$ is independent of $\bar{W}_1$. The broker firm *F* & *H* has invented the derivative "Euler". The holder of a *T*-Euler will, at the time of maturity *T*, obtain the sum

$$\chi = \ln \quad \left[ \{Z(T)\}^2 \right]$$

in SEK. Here $Z(t)$ is the price at time *t* in SEK of the *ACME* stock.

Compute the arbitrage free price (in SEK) at time *t* of a *T*-Euler, given that the price (in SEK) of the *ACME* stock is *z*. The Swedish short rate is denoted by *r*.

**Exercise 6.8** Prove formula (6.50).

**Exercise 6.9** Derive a formula for the value, at *s*, of a forward contract on the *T*-claim *X*, where the forward contract is made at *t*, and $t < s < T$.

## 6.10 Notes

The classics in the field are Black and Scholes (1973), and Merton (1973). The modern martingale approach to arbitrage pricing was developed in Harrison and Kreps (1981), and Harrison and Pliska (1981). A deep study of the connections between (various formulations of) absence of arbitrage and the existence of a martingale measure can be found in Delbaen and Schachermeyer (1994).

For a wealth of information on forward and futures contracts, see Hull (1997) and Duffie (1989). Black's formula was derived in Black (1976). For American options see Barone-Adesi, and Elliott (1991), Geske and Johnson (1984) and Musiela and Rutkowski (1997). The standard reference for optimal stopping problems is Shiryayev (1978), and a very readable exposition can be found in Ø ksendal (1995). Option pricing with stochastic volatility is discussed in Hull and White (1987), and Leland (1985) studies the consequences of introducing transaction costs.

# 7 Completeness and Hedging

## 7.1 Introduction

In the previous chapter we noticed that our derivation of the pricing equation (6.30) was somewhat unsatisfactory, and a major criticism was that we were forced to assume that the derivative asset a priori possessed a price process and actually was traded on the market. In this chapter we will look at arbitrage pricing from a somewhat different point of view, and this alternative approach will have two benefits. Firstly it will allow us to dispose of the annoying Assumption above that the derivative is actually traded, and secondly it will provide us with an explanation of the surprising fact that the simple claims investigated earlier can be given a unique price. For a more detailed discussion see Chapter 10.

We start with a fairly general situation by considering a financial market with a price vector process $S = (S^1, \ldots, S^N)$, governed by an objective probability measure $P$. The process $S$ is as usual interpreted as the price process of the exogenously given underlying assets and we now want to price a contingent $T$-claim $\chi$. We assume that all the underlying assets are traded on the market, but we do not assume that there exists an a priori market (or a price process) for the derivative. To avoid trivialities we also assume that the underlying market is arbitrage free.

**Definition 7.1** *We say that a T-claim* $\chi$ *can be* **replicated** , *alternatively that it is* **reachable** *or* **hedgeable** , *if there exists a self financing portfolio h such that*

$$V^h(T) = \chi, \ P - a.s. \tag{7.1}$$

*In this case we say that h is a* **hedge** *against* $\chi$. *Alternatively, h is called a* **replicating** *or* **hedging** *portfolio. If every contingent claim is reachable we say that the market is* **complete** .

Let us now consider a fixed $T$-claim $\chi$ and let us assume that $\chi$ can be replicated by a portfolio $h$. Then we can make the following mental experiment.

1. Fix a point in time $t$ with $t \leq T$.
2. Suppose that we, at time $t$, possess $V^h(t)$ SEK.
3. We can then use this money to buy the portfolio $h(t)$. If furthermore we follow the portfolio strategy $h$ on the time interval $[t,T]$ this will cost us nothing, since $h$ is self-financing. At time $T$ the value of our portfolio will then be $V^h(T)$ SEK.

4. By the replication Assumption the value, at time $T$, of our portfolio will thus be exactly $\chi$ SEK, regardless of the stochastic price movements over the interval $[t,T]$.
5. From a purely financial point of view, holding the portfolio $h$ is thus equivalent to the holding of the contract $\chi$.
6. The "correct" price of $\chi$ at time $t$ is thus given by $\Pi(t;\chi) = V^h(t)$.

For a hedgeable claim we thus have a natural price process, $\Pi(t;\chi) = V^h(t)$, and we may now ask if this has anything to do with absence of arbitrage.

**Proposition 7.2** *Suppose that the claim $\chi$ can be hedged using the portfolio $h$. Then the only price process $\Pi(t;\chi)$ which is consistent with no arbitrage is given by $\Pi(t;\chi) = V^h(t)$. Furthermore, if $\chi$ can be hedged by $g$ as well as by $h$ then $V^g(t) = V^h(t)$ holds for all $t$ with probability* 1.

**Proof** If at some time $t$ we have $\Pi(t;\chi) < V^h(t)$ then we can make an arbitrage by selling the portfolio short and buying the claim, and vice versa if $\Pi(t;\chi) > V^h(t)$. A similar argument shows that we must have $V^g(t) = V^h(t)$.

# 7.2 Completeness in the Black–Scholes Model

We will now investigate completeness for the generalized Black–Scholes model given by

$$dB(t) = rB(t)dt, \tag{7.2}$$

$$dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))d\bar{W}(t), \tag{7.3}$$

where we assume that $\sigma(t,s) > 0$ for all $(t,s)$. The main result is the following.

**Theorem 7.3** *The model (7.2)–(7.3) is complete.*

The proof of this theorem requires some fairly deep results from probability theory and is thus outside the scope of this book. We will prove a weaker version of the theorem, namely that every **simple** claim can be hedged. This is often quite sufficient for practical purposes, and our proof of the restricted completeness also has the advantage that it gives the replicating portfolio in explicit form. We will use the notational convention $h(t) = [h^0(t), h^*(t)]$ where $h^0$ is the number of bonds in the portfolio, whereas $h^*$ denotes the number of shares in the underlying stock. We thus fix a simple $T$-claim of the form $\chi = \Phi(S(T))$ and we now want to show that this claim can be hedged. Since the formal proof is of the form "consider the following odd construction", we will instead start by presenting a purely heuristic (but good) argument. This argument is, from a formal point of view, only of motivational nature and the logic of it is rather unclear. Since the argument is only heuristic the logical flaws do not matter, since in the end we will in fact present a rigorous statement and a rigorous proof. Before we start the heuristics, let us make more precise what we are looking for. Using Lemma 5.5 we immediately have the following result.

**Lemma 7.4** *Suppose that there exists an adapted process $V$ and an adapted process $u = [u^0, u^*]$ with* *such that*

$$u^0(t) + u^*(t) = 1, \tag{7.4}$$

$$\begin{cases} dV(t) = V(t)\left\{u^0(t)r + u^*(t)\alpha(t,S(t))\right\}dt + V(t)u^*(t)\sigma(t,S(t))d\overline{W}(t), \\ V(T) = \Phi(S(T)). \end{cases} \tag{7.5}$$

*Then the claim* $\chi = \Phi(S(T))$ *can be replicated using u as the relative portfolio. The corresponding value process is given by the process V and the absolute portfolio h is given by*

$$h^0(t) = \frac{u^0(t)V(t)}{B(t)},$$

$$h^*(t) = \frac{u^*(t)V(t)}{S(t)}. \tag{7.6}$$

Our strategy now is to look for a process $V$ and a process $u$ satisfying the conditions above. (7.7)

# Begin Heuristics

We assume what we want to prove, namely that $\chi = \Phi(S(T))$ is indeed replicable, and then we ponder on what the hedging strategy $u$ might look like. Since the $S$-process and (trivially) the $B$-process are Markov processes it seems reasonable to assume that the hedging portfolio is of the form $h(t) = h(t,S(t))$ where, with a slight misuse of notation, the $h$ in the right member of the equality is a deterministic function. Since, furthermore, the value process $V$ (we suppress the superscript $h$) is defined as $V(t) = h^0(t)B(t) + h^*(t)S(t)$ it will also be a function of time and stock price as

$$V(t) = F(t,S(t)), \tag{7.8}$$

where $F$ is some real valued deterministic function which we would like to know more about.

Assume therefore that (7.8) actually holds. Then we may apply the Itô formula to $V$ in order to obtain the $V$-dynamics as

$$dV = \left\{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}\right\}dt + \sigma SF_s d\overline{W}, \tag{7.9}$$

where we have suppressed the fact that $V$ and $S$ are to be evaluated at time $t$, whereas $\alpha$, $\sigma$ and $F$ are to be evaluated at $(t,S(T))$. Now, in order to make (7.9) look more like (7.5) we rewrite (7.9) as

$$dV = V\left\{\frac{F_t + \alpha SF_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V}\right\}dt + V\frac{SF_s}{V}\sigma d\overline{W}. \tag{7.10}$$

Since we have assumed that $\chi$ is replicated by $V$ we see from (7.10) and (7.5) that $u^*$ must be given by

$$u^*(t) = \frac{S(t)F_s(t,S(t))}{F(t,S(t))}, \tag{7.11}$$

(remember that we have assumed that $V(t) = F(t,S(t))$, and if we substitute (7.11)into (7.10) we get

$$dV = V\left\{\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^* \alpha\right\} dt + V u^* \sigma d\overline{W}. \tag{7.12}$$

Comparing this expression to (7.5) we see that the natural choice for $u^0$ is given by

$$u^0 = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF}, \tag{7.13}$$

but we also have to satisfy the requirement $u^0 + u^* = 1$ of (7.4). Using (7.11) and (7.13) this gives us the relation

$$\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} = \frac{F - SF_s}{F}, \tag{7.14}$$

which, after some manipulation, turns out to be the familiar Black–Scholes equation

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0. \tag{7.15}$$

Furthermore, in order to satisfy the relation $F(T,S(T)) = \Phi\,(S(T))$ of (7.5) (remember that we assume that $V(t) = F(t,S(t))$) we must have the boundary value

$$F(T,s) = \Phi\,(s), \text{ for all } \quad s \in R_+. \tag{7.16}$$

# End Heuristics

Since at this point the reader may well be somewhat confused as to the logic of the reasoning, let us try to straighten things out. The logic of the reasoning above is basically as follows.

- We **assumed** that the claim $\chi$ was replicable.
- Using this and some further (reasonable) assumptions we showed that they **implied** that the value process of the replicating portfolio was given as $V(t) = F(t,S(t))$ where $F$ is a solution of the Black–Scholes equation.

This is of course not at all what we wish to achieve. What we want to do is to **prove** that $\chi$ really can be replicated. In order to do this we put the entire argument above within a logical parenthesis and formally disregard it. We then have the following result.

**Theorem 7.5** *Consider the market (7.2)–(7.3), and a contingent claim of the form* $\chi = \Phi(S(T))$. *Define F as the solution to the boundary value problem*

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF = 0, \\ F(T,s) = \Phi(s). \end{cases}$$

*Then* $\chi$ *can be replicated by the relative portfolio*

(7.17)

$$u^0(t) = \frac{F(t,S(t)) - S(t)F_s(t,S(t))}{F(t,S(t))},$$

(7.18)

$$u^*(t) = \frac{S(t)F_s(t,S(t))}{F(t,S(t))}.$$

(7.19)

*The corresponding absolute portfolio is given by*

$$h^0(t) = \frac{F(t,S(t)) - S(t)F_s(t,S(t))}{B(t)},$$

(7.20)

$$h^*(t) = F_s(t,S(t)),$$

*and the value process* $V^h$ *is given by*

(7.21)

$$V^h(t) = F(t,S(t)).$$

(7.22)

**Proof** Applying the Itô formula to the process $V(t)$ defined by (7.22) and performing exactly the same calculations as in the heuristic argument above, will show that we can apply Lemma 7.4.

The result above gives us an explanation of the surprising facts that there actually exists a **unique** price for a derivative asset in the Black–Scholes model and that this price does not depend on any particular assumptions about individual preferences. The arbitrage free price of a derivative asset is uniquely determined simply because in this model the derivative is superfluous. It can always be replaced by a corresponding "synthetic" derivative in terms of a replicating portfolio.

Since the replication is done with $P$-probability 1, we also see that if a contingent claim $\chi$ is replicated under $P$ by a portfolio $h$ and if $P^*$ is some other probability measure such that $P$ and $P^*$ assign probability 1 to exactly the same events (such measures $P$ and $P^*$ are said to be **equivalent** ), then $h$ will replicate $\chi$ also under the measure $P^*$. Thus the pricing formula for a certain claim will be exactly the same for all measures which are equivalent to $P$. It is a well known fact (the Girsanov theorem) in the theory of SDEs that if we change the measure from $P$ to some other equivalent measure, this will change the drift in the SDE, but the diffusion term will be unaffected. Thus the drift will play no part in the pricing equation, which explains why $\alpha$ does not appear in the Black–Scholes equation.

Let us now list some popular claims and see which of them will fall into the framework above.

$$\chi = \max\,[S(T) - K, 0] \quad \text{(European call option)}$$

$$\chi = S(T) - K \quad \text{(Forward contract)} \tag{7.23}$$

$$\chi = \max\left[\frac{1}{T}\int_0^T S(t)dt - K, 0\right] \quad \text{(Asian option)} \tag{7.24}$$

$$\chi = S(T) - \inf_{0 \le t \le T} S(t) \quad \text{(Lookback contract)} \tag{7.25}$$

(7.26)

We know from Theorem 7.3 that in fact all of the claims above can be replicated. For general claims this is, however, only an abstract existence result and we have no guarantee of obtaining the replicating portfolio in an explicit form. The point of Theorem 7.5 is precisely that, by restricting ourselves to **simple** claims, i.e. claims of the form $\chi = \Phi(S(T))$, we obtain an explicit formula for the hedging portfolio.

It is clear that the European call as well as the forward contract above are simple claims, and we may thus apply Theorem 7.5. The Asian option (also called a mean value option) as well as the lookback present harder problems since neither of these claims is simple. Instead of just being functions of the value of $S$ at time $T$ we see that the claims depend on the entire $S$-trajectory over the interval $[0,T]$. Thus, while we know that there exist hedging portfolios for both these claims, we have presently no obvious way of determining the shape of these portfolios.

It is in fact quite hard to determine the hedging portfolio for the lookback, but the Asian option belongs to a class of contracts for which we can give a fairly explicit representation of the replicating portfolio, using very much the same technique as in Theorem 7.5.

**Proposition 7.6** *Consider the model*

$$dB(t) = rB(t)dt,$$

$$dS(t) = S(t)\alpha(t,S(t))dt + S(t)\sigma(t,S(t))d\bar{W}(t), \tag{7.27}$$

*and let $\chi$ be a T-claim of the form*

(7.28)

$$\chi = \Phi\,(S(T), \quad Z(T)),$$

*where the process Z is defined by*

(7.29)

$$Z(t) = \int_0^t g(u,S(u))du,$$

*for some choice of the deterministic function g. Then $\chi$ can be replicated using a relative portfolio given by*

(7.30)

$$u^0(t) = \frac{F(t,S(t),Z(t)) - S(t)F_s(t,S(t),Z(t))}{F(t,S(t),Z(t))},$$

(7.31)

$$u^*(t) = \frac{S(t)F_s(t,S(t),Z(t))}{F(t,S(t),Z(t))},$$

*where F is the solution to the boundary value problem*

(7.32)

$$\begin{cases} F_t + srF_s + \frac{1}{2}s^2\sigma^2F_{ss} + gF_z - rF = 0, \\ F(T,s,z) = \Phi(s,z). \end{cases}$$

(7.33)

*The corresponding value process V is given by* $V(t) = F(t,S(t),Z(t))$, *and F has the stochastic representation*

$$F(t,s,z) = e^{-r(T-t)}E^Q_{t,s,z}[\Phi(S(T),Z(T))],$$

*where the Q-dynamics are given by*

(7.34)

$$dS(u) = rS(u)du + S(u)\sigma(u,S(u))dW(u),$$

$$S(t) = s, \tag{7.35}$$

$$dZ(u) = g(u,S(u))du, \tag{7.36}$$

$$Z(t) = z. \tag{7.37}$$

(7.38)

**Proof** The proof is left as an exercise for the reader. Use the same technique as in the proof of Theorem 7.5.

Again we see that the arbitrage free price of a contingent claim is given as the expected value of the claim discounted to the present time. Here, as before, the expected value is to be calculated using the martingale measure $Q$ instead of the objective probability measure $P$. As we have said before, this general structure of arbitrage free pricing holds in much more general situations, and as a rule of thumb one can view the martingale measure $Q$ as being defined by the property that all **traded** underlying assets have $r$ as the rate of return under $Q$. It is important to stress that it is only traded assets which will have s $r$ as the rate of return under $Q$. For models with nontraded underlying objects we have a completely different situation, which we will encounter below.

# 7.3 Completeness—Absence of Arbitrage

In this section we will give some general rules of thumb for quickly determining whether a certain model is complete and/or free of arbitrage. The arguments will be purely heuristic.

Let us consider a model with $M$ traded underlying assets **plus** the risk free asset (i.e. totally $M + 1$ assets). We assume that the price processes of the underlying assets are driven by $R$ "random sources". We cannot give a precise definition of what constitutes a "random source" here, but the typical example is a driving Wiener process. If, for example, we have five independent Wiener processes driving our prices, then $R = 5$. Another example of a random source would be a counting process such as a Poisson process. In this context it is important to note that if the prices are driven by a point process with different jump sizes then the appropriate number of random sources equals the number of different jump sizes.

When discussing completeness and absence of arbitrage it is important to realize that these concepts work in opposite directions. Let the number of random sources $R$ be fixed. Then every new underlying asset added to the model (without increasing $R$) will of course give us a potential opportunity of creating an arbitrage portfolio, so in order to have an arbitrage free market the number $M$ of underlying assets must be small in comparison to the number of random sources $R$.

On the other hand we see that every new underlying asset added to the model gives us new possibilities of replicating a given contingent claim, so completeness requires $M$ to be great in comparison to $R$.

We cannot formulate and prove a precise result here, but the following rule of thumb, or "meta-theorem", is nevertheless extremely useful. In concrete cases it can in fact be given a precise formulation and a precise proof. We will later use the meta-theorem when dealing with problems connected with nontraded underlying assets in general and interest rate theory in particular.

**Meta-theorem 7.3.1** *Let M denote the number of underlying* **traded** *assets in the model* **excluding** *the risk free asset, and let R denote the number of random sources. Generically we then have the following relations.*

1. *The model is arbitrage free if and only if $M \leq R$.*
2. *The model is complete if and only if $M \geq R$.*
3. *The model is complete and arbitrage free if and only if $M = R$.*

As an example we take the Black–Scholes model, where we have one underlying asset $S$ plus the risk free asset so $M = 1$. We have one driving Wiener process, giving us $R = 1$, so in fact $M = R$. Using the meta-theorem above we thus expect the Black–Scholes model to be arbitrage free as well as complete and this is indeed the case.

## 7.4 Exercises

**Exercise 7.1** Consider a model for the stock market where the short rate of interest $r$ is a deterministic constant. We focus on a particular stock with price process $S$. Under the objective probability measure $P$ we have the following dynamics for the price process.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + \delta S(t-)dN(t).$$

Here $W$ is a standard Wiener process whereas $N$ is a Poisson process with intensity $\lambda$. We assume that $\alpha$, $\sigma$, $\delta$ and $\lambda$ are known to us. The $dN$ term is to be interpreted in the following way.

- Between the jump times of the Poisson process $N$, the $S$-process behaves just like ordinary geometric Brownian motion.
- If $N$ has a jump at time $t$ this induces $S$ to have a jump at time $t$. The size of the $S$-jump is given by

$$S(t) - S(t-) = \delta \cdot S(t-).$$

Discuss the following questions.

(a) Is the model free of arbitrage?
(b) Is the model complete?
(c) Is there a unique arbitrage free price for, say, a European call option?
(d) Suppose that you want to replicate a European call option maturing in January 1999. Is it posssible (theoretically) to replicate this asset by a portfolio consisting of bonds, the underlying stock and European call option maturing in December 2001?

**Exercise 7.2** Use the Feynman–Kač technique in order to derive a risk neutral valuation formula in connection with Proposition 7.6.

**Exercise 7.3** The fairly unknown company *F & H INC.* has blessed the market with a new derivative, "the Mean". With "effective period" given by $[T_1,T_2]$ the holder of a Mean contract will, at the date of maturity $T_2$, obtain the amount

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S(u)du.$$

Determine the arbitrage free price, at time $t$, of the Mean contract. Assume that you live in a standard Black–Scholes world, and that $t < T_1$.

**Exercise 7.4** Consider the standard Black–Scholes model, and $n$ different simple contingent claims with contract functions $\Phi_1, \ldots, \Phi_n$. Let

$$V = \sum_{i=1}^{n} h_i(t)S_i(t)$$

denote the value process of a self-financing, Markovian (see Definition 5.2) portfolio. Because of the Markovian Assumption, $V$ will be of the form $V(t,S(t))$. Show that $V$ satisfies the Black–Scholes equation.

# 7.5 Notes

Completeness is mathematically closely related to (rather deep) results about the possibility of representing martingales as sums of stochastic integrals. Using this connection, it can be shown that the market is complete if and only if the martingale measure is unique. See Harrison and Pliska (1981), Musiela and Rutkowski (1997).

# 8 Parity Relations and Delta Hedging

## 8.1 Parity Relations

Consider the standard Black–Scholes model. As we know from general theory (Theorem 7.3) this model allows us to replicate any contingent claim using a portfolio based on the underlying asset and the risk free asset. For a nontrivial claim the structure of the hedging portfolio is typically quite complicated, and in particular it is a portfolio which is continuously rebalanced. For practical purposes this continuous rebalancing presents a problem because in real life trading does have a cost. For managerial purposes it would be much nicer if we could replicate a given claim with a portfolio which did not have to be rebalanced, in other words a portfolio which is **constant** over time. Such a portfolio is known as a **buy-and-hold** portfolio.

If we insist on using only $B$ and $S$ in our replicating portfolio we cannot replicate any interesting claims using constant portfolios, but if we allow ourselves to include some derivative, like a European call option, in our hedging portfolio, then life becomes much simpler, and the basic result we will use is the following trivial linear property of pricing.

**Proposition 8.1** *Let* $\Phi$ *and* $\Psi$ *be contract functions for the T-claims* $\chi = \Phi(S(T))$ *and* $Y = \Psi(S(T))$*. Then for any real numbers* $\alpha$ *and* $\beta$ *we have the following price relation.*

$$\Pi(t; \alpha\Phi + \beta\Psi) = \alpha\Pi(t; \Phi) + \beta\Pi(t; \Psi).$$

(8.1)

**Proof** This follows immediately from the risk neutral valuation formula (6.38) and the linear property of mathematical expectation.

To set notation let $c(t,s;K,T,r,\sigma)$ and $p(t,s;K,T,r,\sigma)$ denote the price at time $t$ given $S(t) = s$ of a European call option and a European put option respectively. In both cases $T$ denotes the time of maturity, $K$ the strike price, whereas $r$ and $\sigma$ indicate the dependence on model parameters. From time to time we will freely suppress one or more of the variables $(t,s,K,T,r,\sigma)$. Let us furthermore consider the following "basic" contract functions.

$$\Phi_S(x) = x,$$

(8.2)

$$\Phi_B(x) \equiv 1,$$

(8.3)

$$\Phi_{C,K}(x) = \max\left[x - K, 0\right].$$

(8.4)

The corresponding claims at the time of maturity give us one share of the stock, \$1, and one European call with strike price $K$ respectively. For these claims the prices are given by

$$\Pi(t;\Phi_S) = S(t),$$

$$\Pi(t;\Phi_B) = e^{-r(T-t)}, \tag{8.5}$$

$$\Pi(t;\Phi_{C,K}) = c(t,S(t);K,T). \tag{8.6}$$

(8.7)

Let us now fix a time of maturity $T$ and a $T$-claim $\chi$ of the form $\chi = \Phi(S(T))$, i.e. a simple claim. It is now clear that if $\Phi$ is a linear combination of the basic contracts above, i.e. if we have

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i},$$

then we may price $\Phi$ in terms the prices of the basic contracts as

(8.8)

$$\Pi(t;\Phi) = \alpha\Pi(t;\Phi_S) + \beta\Pi(t;\Phi_B) + \sum_{i=1}^{n} \gamma_i \Pi(t;\Phi_{C,K_i}).$$

(8.9)

Note also that in this case we may replicate the claim $\Phi$ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a so called "buy-and hold" portfolio. More precisely the replicating portfolio consists of

- $\alpha$ shares of the underlying stock,
- $\beta$ zero coupon $T$-bonds with face value \$1,
- $\gamma_i$ European call options with strike price $K_i$, all maturing at $T$.

The result above is of course interesting only if there is a reasonably large class of contracts which in fact can be written as linear combinations of the basic contracts given by (8.2), (8.3) and (8.4). This is indeed the case, and as a first example we consider the European put option with strike price $K$, for which the contract function $\Phi_{P,K}$ is defined by

$$\Phi_{P,K}(x) = \max[K - x, 0].$$

It is now easy to see (draw a figure!) that

(8.10)

$$\Phi_{P,K} = K\Phi_B + \Phi_{C,K} - \Phi_S,$$

so we have the following so called put–call parity relation.

**Proposition 8.2 (Pu–call parity)** *Consider a European call and a European put, both with strike price K and time of maturity T. Denoting the corresponding pricing functions by c(t,s) and p(t,s), we have the following relation.*

$$p(t,s) = Ke^{-r(T-t)} + c(t,s) - s.$$

(8.11)

*In particular the put option can be replicated with a constant (over time) portfolio consisting of a long position in a zero coupon T-bond with face value K, a* *long position in a European call option and a short position in one share of the underlying stock.*

It is now natural to pose the following more general question. Which contracts can be replicated in this way using a constant portfolio consisting of bonds, call options and the underlying stock? The answer is very pleasing.

**Proposition 8.3** *Fix an arbitrary continuous contract function* $\Phi$ *with compact support. Then the corresponding contract can be replicated with arbitrary precision (in sup-norm) using a constant portfolio consisting only of bonds, call options and the underlying stock.*

**Proof** It is easily seen that any affine function can be written as a linear combination of the basic contract functions. The result now follows from the fact that any continuous function with compact support can be approximated uniformly by a piecewise linear function.

## 8.2 The Greeks

Let $P(t,s)$ denote the pricing function at time $t$ for a portfolio based on a **single underlying asset** with price process $S_t$. The portfolio can thus consist of a position in the underlying asset itself, as well as positions in various options written on the underlying asset. For practical purposes it is often of vital importance to have a grip on the sensitivity of $P$ with respect to the following.

1. Price changes of the underlying asset.
2. Changes in the model parameters.

In case 1 above we want to obtain a measure of our risk exposure, i.e. how the value of our portfolio (consisting of stock and derivatives) will change given a certain change in the underlying price. At first glance case 2 seems self-contradictory, since a model parameter is by definition a given constant, and thus it cannot possibly change within the given model. This case is therefore not one of risk exposure but rather one of sensitivity with respect to misspecifications of the model parameters.

We introduce some standard notation.

**Definition 8.4**

$$\Delta = \frac{\partial P}{\partial s}, \tag{8.12}$$

$$\Gamma = \frac{\partial^2 P}{\partial s^2}, \tag{8.13}$$

$$\rho = \frac{\partial P}{\partial r}, \tag{8.14}$$

$$\Theta = \frac{\partial P}{\partial t}, \tag{8.15}$$

$$\nu = \frac{\partial P}{\partial \sigma}. \tag{8.16}$$

Fig. 8.1 Delta For a European Call

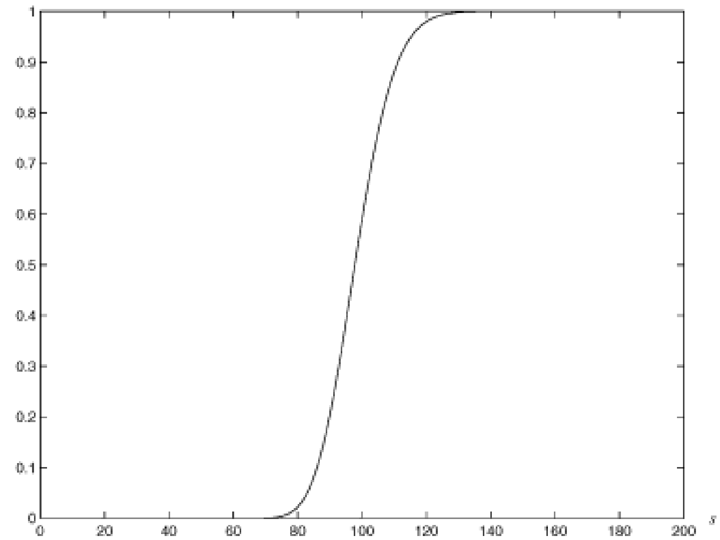
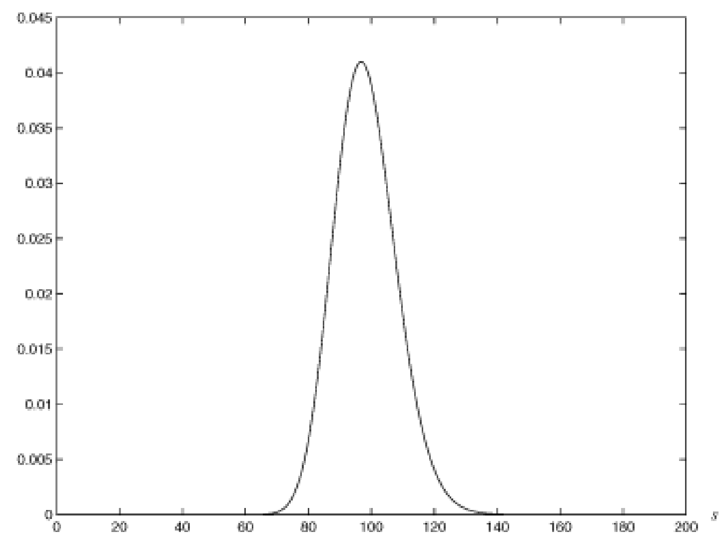



Fig. 8.2 Gamma For a European Call




All these sensitivity measures are known as “the greeks”. This includes $\mathcal{V}$, which in this case is given the Anglo-Hellenic pronounciation “vega”. A portfolio which is insensitive w.r.t. small changes in one of the parameters above is said to be **neutral** , and formally this means that the corresponding greek equals zero. A portfolio with zero delta is said to be **delta neutral** , and correspondingly for the other greeks. In the next section we will study various hedging schemes, based upon the greeks, but first we present the basic formulas for the case of a call option. See Figs 8.1–8.5 for graphs of the greeks as functions of the underlying stock price.

Fig. 8.3 Vega For a European Call

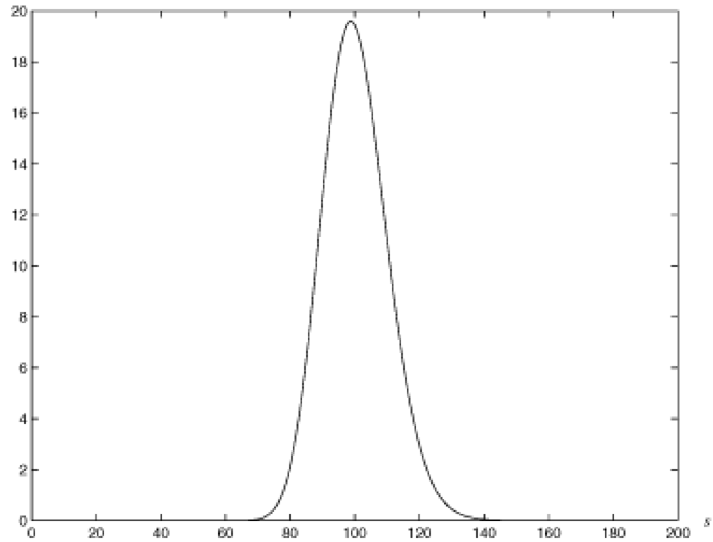
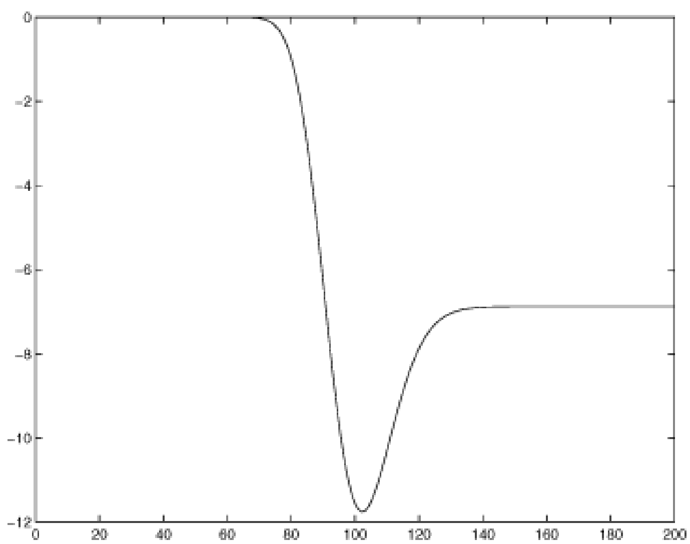



Fig. 8.4 Theta For a European Call




**Proposition 8.5** *For a European call with strike price K and time of maturity T we have the following relations, with notation as in the Black–Scholes formula. The letter* $\varphi$ *denotes the density function of the* N[0,1] *distribution.*

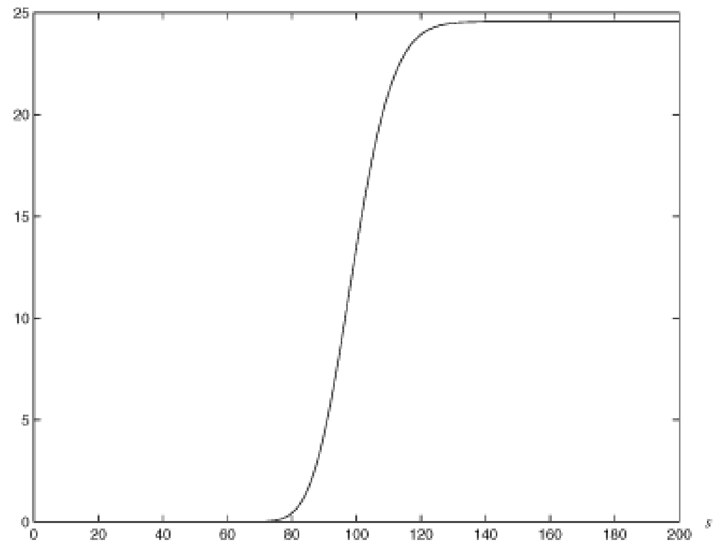
$$\Delta = N(d_1),$$

$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}, \tag{8.17}$$

$$\rho = K(T-t)e^{-r(T-t)}N(d_2), \tag{8.18}$$

(8.19)

Fig. 8.5 Rho For a European Call




$$\Theta = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2), \tag{8.20}$$

$$\nu = s\varphi(d_1)\sqrt{T-t}. \tag{8.21}$$

**Proof** Use the Black–Scholes formula (6.46) and take derivatives. The (brave) reader is invited to carry this out in detail. The calculations are sometimes quite messy.

## 8.3 Delta and Gamma Hedging

As in the previous section, let us consider a given portfolio with pricing function $P(t,s)$. The object is to immunize this portfolio against small changes in the underlying asset price $s$. If the portfolio already is delta neutral, i.e. if

$$\Delta_P = \frac{\partial P}{\partial s} = 0,$$

then we are done, but what can we do in the more interesting case when $\Delta_P \neq 0$? One possibility is of course to sell the entire portfolio, and invest the sum thus obtained in the bank, but this is in most cases neither practically feasible, nor preferable.

A more interesting Idea is to add a derivative (e.g. an option or the underlying asset itself) to the portfolio. Since the price of a derivative is perfectly correlated with the underlying asset price, we should be able to balance the derivative against the portfolio in such a way that the adjusted portfolio becomes delta neutral. The reader will recognize this argument from the derivation of the Black–Scholes PDE, and the formal argument is as follows.

We denote the pricing function of the chosen derivative by $F(t,s)$, and $x$ denotes the number of units of the derivative which we will add to the a priori given portfolio. The value $V$ of the adjusted portfolio is then given by

$$V(t,s) = P(t,s) + x \cdot F(t,s). \tag{8.22}$$

In order to make this portfolio delta neutral we have to choose $x$ such that $\frac{\partial V}{\partial s} = 0$, and this gives us the equation

$$\frac{\partial P}{\partial s} + x\frac{\partial F}{\partial s} = 0,$$

which, with obvious notation, has the solution

$$x = -\frac{\Delta_P}{\Delta_F}. \tag{8.23}$$

**Example 8.6** Let us assume that we have sold a particular derivative with pricing function $F(t,s)$, and that we wish to hedge it using the underlying asset itself. In (8.22) we now have $P = -1 \cdot F$, whereas $F$ is replaced by $s$, and we get the equation

$$\frac{\partial}{\partial s}[-F(t,s) + x \cdot s] = 0,$$

with the solution

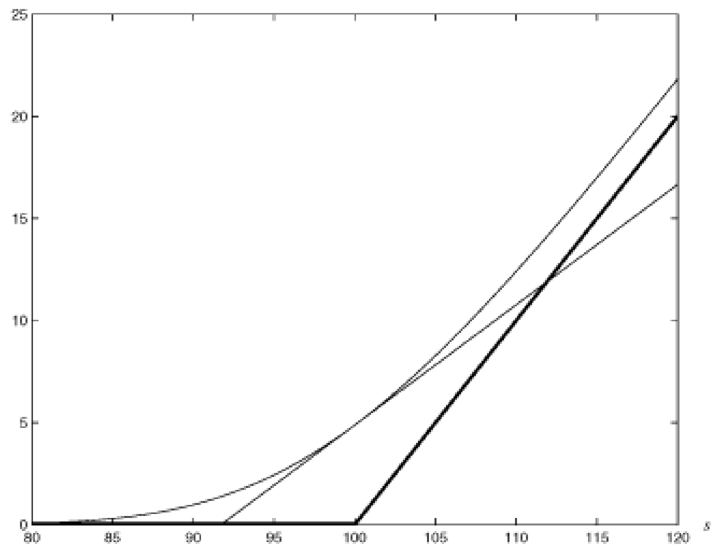
$$x = \Delta_F = \frac{\partial F(t,s)}{\partial s}.$$

We thus see that the delta of a derivative gives us the number of units of the underlying stock that is needed in order to hedge the derivative.

It is important to note that a delta hedge only works well for small changes in the underlying price, and thus only for a short time. In Example 8.6 above, what we did was to approximate the pricing function $F(t,s)$ with its tangent, and in Fig. 8.6 this is illustrated for the case when $F$ is the pricing function of a European call option. $\Delta_F$ equals the slope of the tangent. In Fig. 8.1 we have a graph of the delta of a European call, as a function of the underlying stock price. As time goes by the value of $s$ (and $t$) will change, and thus we will be using an old, incorrect value of $\Delta$. What is done in practice is to perform a **discrete rebalanced delta hedge** , which for the example above can be done along the following lines.

- Sell one unit of the derivative at time $t = 0$ at the price $F(0,s)$.
- Compute $\Delta$ and buy $\Delta$ shares. Use the income from the sale of the derivative, and if necessary borrow money from the bank.
- Wait one day (week, minute, second). The stock price has now changed and your old $\Delta$ is no longer correct.
- Compute the new value of $\Delta$ and adjust your stock holdings accordingly. Balance your account by borrowing from or lending to the bank.
- Repeat this procedure until the exercise time $T$.
- In this way the value of your stock and money holdings will approximately equal the value of the derivative.

Fig. 8.6 Linear Approximation Of a European Call




It is in fact not hard to prove (see the exercises) the following asymptotic result.

**Proposition 8.7** *In a continuously rebalanced delta hedge, the value of the stock and money holdings will replicate the value of the derivative.*

In a (discrete) scheme of the kind above we face a dilemma concerning the frequency of the rebalancing points in time. If we rebalance often, we will have a very good hedge, but we will also suffer from high transaction costs. The reason why we have to rebalance is that delta changes as the underlying price changes, and a measure of the sensitivity of $\Delta$ with respect to $s$ is of course given by

$$\Gamma = \frac{\partial \Delta}{\partial s} = \frac{\partial^2 P}{\partial s^2}$$

. See Fig. 8.2 for a graph of the gamma of a European call. If gamma is high we have to rebalance often, whereas a low gamma will allow us to keep the delta hedge for a longer period. It is thus preferable to form a portfolio which, apart from being delta neutral, is also **gamma neutral** .

In order to analyze this in some generality, let us again consider an a priori given portfolio with price function $P(t,s)$. For future use we state the following trivial but important facts.

**Lemma 8.8** *For the underlying stock, the delta and gamma are given by*

$$\begin{aligned} \Delta_S &= 1, \\ \Gamma_S &= 0. \end{aligned}$$

From the fact that the gamma of the underlying stock equals zero, it follows that we cannot use the stock itself in order to change the gamma of the portfolio. Since we want the adjusted portfolio to be both delta and gamma neutral, it is also obvious that we need two different derivatives in the hedge. Let us thus fix two derivatives, e.g. two call options with different exercise prices or different times of maturity, with pricing functions $F$ and $G$. We denote the number of units of the derivatives by $x_F$ and $x_G$ respectively, and the value of the hedged portfolio is now given by

$$V = P(t,s) + x_F \cdot F(t,s) + x_G \cdot G(t,s).$$

In order to make this portfolio both delta and gamma neutral we have to choose $x_F$ and $x_G$ such that the equations

$$\frac{\partial V}{\partial s} = 0,$$

$$\frac{\partial^2 V}{\partial s^2} = 0,$$

are satisfied. With obvious notation we thus obtain the system

$$\Delta_P + x_F \cdot \Delta_F + x_G \cdot \Delta_G = 0, \tag{8.24}$$

$$\Gamma_P + x_F \cdot \Gamma_F + x_G \cdot \Gamma_G = 0, \tag{8.25}$$

which can easily be solved.

It is natural, and very tempting, to construct a delta and gamma neutral hedge by the following two step procedure.

1. Choose $x_F$ such that the portfolio consisting of $P$ and $F$ is delta neutral. This portfolio will generally not be gamma neutral.
2. Now add the derivative $G$ in order to make the portfolio gamma neutral.

The problem with this scheme is that the second step in general will destroy the delta neutrality obtained by the first step. In this context we may, however, use the fact that the stock itself has zero gamma and we can thus modify the scheme as follows.

1. Choose $x_F$ such that the portfolio consisting of $P$ and $F$ is gamma neutral. This portfolio will generally not be delta neutral.
2. Now add the underlying stock in order to make the portfolio delta neutral.

Formally the value of the hedged portfolio will now be given by

$$V = P + x_F \cdot F + x_S \cdot s$$

and, using the lemma above, we obtain the following system.

$$\Delta_P + x_F \cdot \Delta_F + x_S = 0, \tag{8.26}$$

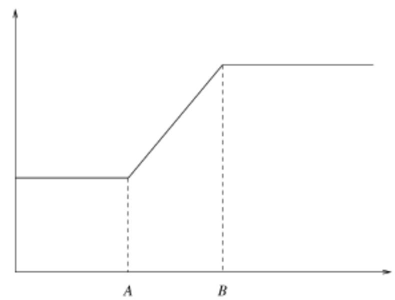
$$\Gamma_P + x_F \cdot \Gamma_F = 0. \tag{8.27}$$

This system is triangular, and thus much simpler than the system (8.24)–(8.25). The solution is given by

$$x_F = -\frac{\Gamma_P}{\Gamma_F},$$

$$x_S = \frac{\Delta_F \cdot \Gamma_P}{\Gamma_F} - \Delta_P.$$

Fig. 8.7.




Using the technique described above one can easily derive hedging schemes in order to make a given portfolio neutral with respect to any of the greeks above. This is, however, left to the reader.

## 8.4 Exercises

**Exercise 8.1** Consider the standard Black–Scholes model. Fix the time of maturity $T$ and consider the following $T$-claim $\chi$.

$$\chi = \begin{cases} K & \text{if } S(T) \leq A \\ K + A - S(T) & \text{if } A < S(T) < K + A \\ 0 & \text{if } S(T) > K + A. \end{cases} \tag{8.28}$$

This contract can be replicated using a portfolio, consisting solely of bonds, stock and European call options, which is constant over time. Determine this portfolio as well as the arbitrage free price of the contract.

**Exercise 8.2** The setup is the same as in the previous exercise. Here the contract is a so called **straddle** , defined by

$$\chi = \begin{cases} K - S(T) & \text{if } 0 < S(T) \leq K \\ S(T) - K & \text{if } K < S(T). \end{cases} \tag{8.29}$$

Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

**Exercise 8.3** The setup is the same as in the previous exercises. We will now study a so called “bull spread” (see Fig. 8.7). With this contract we can, to a limited extent, take advantage of an increase in the market price while being protected from a decrease. The contract is defined by

$$\chi = \begin{cases} B & \text{if } S(T) > B \\ S(T) & \text{if } A \leq S(T) \leq B \\ A & \text{if } S(T) < A. \end{cases} \tag{8.30}$$

We have of course the relation $A < B$. Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

**Exercise 8.4** The setup and the problem are the same as in the previous exercises. The contract is defined by

$$\chi \begin{cases} 0 & \text{if } S(T) < A \\ S(T) - A & \text{if } A \le S(T) \le B \\ C - S(T) & \text{if } B \le S(T) \le C \\ 0 & \text{if } S(T) > C. \end{cases}$$

8.31

By definition the point $C$ divides the interval $[A,C]$ in the middle, i.e $B = \frac{A+C}{2}$.

**Exercise 8.5** Suppose that you have a portfolio $P$ with $\Delta_P = 2$ and $\Gamma_P = 3$. You want to make this portfolio delta and gamma neutral by using two derivatives $F$ and $G$, with $\Delta_F = -1$, $\Gamma_F = 2$, $\Delta_G = 5$ and $\Gamma_G = -2$. Compute the hedge.

**Exercise 8.6** Consider the same situation as above, with the difference that now you want to use the underlying $S$ instead of $G$. Construct the hedge according to the two step scheme descibed in Section 8.3.

**Exercise 8.7** Prove Proposition 8.7 by comparing the stock holdings in the continuously rebalanced portfolio to the replicating portfolio in Theorem 7.5 of the previous chapter.

**Exercise 8.8** Consider a self-financing Markovian portfolio (in continuous time) containing various derivatives of the single underlying asset in the Black–Scholes model. Denote the value (pricing function) of the portfolio by $P(t,s)$. Show that the following relation must hold between the various greeks of $P$.

$$\Theta_P + rs\Delta_P + \frac{1}{2}\sigma^2 s^2 \Gamma_P = rP.$$

**Hint:** Use Exercise 7.4.

**Exercise 8.9** Use the result in the previous exercise to show that if the portfolio is both delta and gamma neutral, then it replicates the risk free asset, i.e. it has a risk free rate of return which is equal to the short rate $r$.

**Exercise 8.10** Show that for a European put option the delta and gamma are given by

$$\Delta = N[d_1] - 1,$$
$$\Gamma = \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}.$$

**Hint:** Use put–call parity.

**Exercise 8.11** Take as given the usual portfolio $P$, and investigate how you can hedge it in order to make it both delta and vega neutral.

# 9 Several Underlying Assets

## 9.1 Introduction

In this chapter we will generalize the Black–Scholes model to the case where, apart from the risk free asset, we have several underlying assets. We thus assume that we have $n$ a priori given assets ("stocks") with price processes $S_1(t),\dots,$ $S_n(t)$. The entire asset price vector is denoted by $S(t)$, and in matrix notation we will write it as a column vector

$$S(t) = \begin{bmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{bmatrix}.$$

The main problems are those of pricing and hedging contingent claims of the form

$$\chi = \Phi\,(S(T)),$$

where $T$ as usual is a fixed exercise time.

The first problem to be attacked is how to construct a "reasonable" mathematical model for the dynamics of the asset price vector $S$, and in this context we have two demands. We of course want the model to be free of arbitrage possibilities, and we also want the model to be such that we have a unique arbitrage free price process $\Pi\;(t;\chi)$ for any given claim $\chi$.

From the Meta-theorem 7.3.1 we know that we may generically expect absence of arbitrage if we have at least as many sources of randomness as we have underlying assets, so it is natural to demand that the price vector $S$ should be driven by at least $n$ independent Wiener processes.

If, on the other hand, we want a unique price process for every claim, then we need a complete market model, and according to Meta-theorem 7.3.1 this will only occur if we have at least as many assets as we have sources of randomness. In order to obtain a nicely behaved model we are thus forced to model the stock price dynamics using exactly $n$ independent Wiener processes, and we now go on to specify the formal model.

**Assumption 9.1.1** *We assume the following.*

- *Under the objective probability measure P, the S-dynamics are given by*

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t)\sum_{j=1}^{n}\sigma_{ij}d\bar{W}_j(t), \tag{9.1}$$

*for* $i = 1, \ldots, n$. *Here* $W_1, \ldots, W_n$ *are independent P-Wiener processes.*

- *The coefficients* $\alpha_i$ *and* $\sigma_{ij}$ *above are assumed to be known constants.*
- *The* **volatility matrix**

$$\sigma = \{\sigma_{ij}\}_{i,j=1}^{n}$$

  *is nonsingular.*
- *We have the standard risk free asset with price process B, where*

$$dB(t) = rB(t)dt. \tag{9.2}$$

The assumption that the coefficients are constants is made for ease of exposition. Later on we will see that we may allow the coefficients to be functions of current time and current stock prices, i.e. $\alpha_i = \alpha_i(t, S(t))$, $\sigma_{ij} = \sigma_{ij}(t, S(t))$.

In the sequel we will let $W(t)$ denote the column vector

$$\bar{W}(t) = \begin{bmatrix} \bar{W}_1(t) \\ \vdots \\ \bar{W}_n(t) \end{bmatrix}$$

and it will be convenient to define the row vector $\sigma_i$ as the $i$th row of the volatility matrix $\sigma$, i.e.

$$\sigma_i = [\sigma_{i1}, \ldots, \sigma_{in}].$$

With this notation we may write the stock price dynamics more compactly as

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t)\sigma_i d\bar{W}(t). \tag{9.3}$$

It is in fact possible to write the $S$-dynamics even more compactly. For any $n$-vector $x = (x_1, \ldots, x_n)$ we let $D[x]$ denote the diagonal matrix

$$D[x] = \begin{bmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n \end{bmatrix}$$

and we let $\alpha$ denote the column vector

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}. \tag{9.4}$$

Using this notation we can write the $S$-dynamics as

$$dS(t) = D[S(t)]\alpha dt + D[S(t)]\sigma d\bar{W}(t). \tag{9.5}$$

## 9.2 Pricing

We take the market model above as given and we consider a fixed $T$-claim of the form

$$\Phi\,(S(T)).$$

The problem is to find the arbitrage free price process $\Pi\,(t;\chi)$ for $\chi$, and we will do this by using a slight variation of the technique for the one-dimensional case. As before we start by assuming that there actually **is** a market price process for the claim, and that the price process is of the form

$$\Pi\,(t;\chi) = F(t,S(t)),$$

for some deterministic function

$$F{:}R_+ \times R^n \rightarrow R.$$

Our problem is to find out what $F$ must look like, in order not to introduce any arbitrage possibilities, if we are allowed to trade in the derivative as well as in the underlying assets. More precisely, we want the market $[S_1(t),\ldots,S_n(t),\ \Pi\,(t;\chi)]$ to be free of arbitrage, and the basic scheme is as follows.

- Take the model for the underlying assets, the contract function $\Phi$, and the pricing function $F$ as given.
- Form a self-financing portfolio, based on $S_1, \ldots, S_n, B$ and $F$. Since we have $n + 2$ assets, and the portfolio weights must add to unity, this will give us $n + 1$ degrees of freedom in our choice of weights.
- Choose the portfolio weights such that the driving Wiener processes are canceled in the portfolio, thus leaving us with portfolio dynamics of the form

  $$dV(t) = V(t)k(t)dt.$$

  Since we have $n$ driving Wiener processes this will "use up" $n$ degrees of freedom.
- Use the remaining degree of freedom in order to force the value dynamics to be of the form

  $$dV(t) = (r + \beta)V(t)dt,$$

  where $\beta$ is some fixed nonzero real number. In the equation above we think of $\beta$ as being a positive real number, so we are in effect trying to "beat the risk free asset" $B$ by constructing a synthetic risk free asset with higher rate of return than the money account. It turns out that this is technically possible if and only if a certain matrix is nonsingular.
- Since we assume that the market is free of arbitrage possibilities, it is impossible to beat the risk free asset in the way described above. Therefore, the matrix mentioned above has to be singular.

- The singularity condition of the matrix leads to a PDE for the pricing function $F$, and the solution of this PDE is thus the unique arbitrage free pricing function for the claim $\chi$.

To put these ideas into action we start by computing the price dynamics of the derivative. The multidimensional Itô formula gives us (see Remark 3.7.1), after some reshuffling,

$$dF = F \cdot \alpha_F dt + F \cdot \sigma_F d\bar{W}, \tag{9.6}$$

where

$$\alpha_F(t) = \frac{1}{F}\left[F_t + \sum_1^n \alpha_i S_i F_i + \frac{1}{2} tr\left\{\sigma^* D[S] F_{ss} D[S] \sigma\right\}\right], \tag{9.7}$$

$$\sigma_F(t) = \frac{1}{F} \sum_1^n S_i F_i \sigma_i. \tag{9.8}$$

Here the arguments $t$ and $S(t)$ have been suppressed, and we have used the notation

$$\begin{aligned} F_t &= \frac{\partial F}{\partial t}(t, S(t)), \\ F_i &= \frac{\partial F}{\partial s_i}(t, S(t)), \\ F_{ss} &= \left\{\frac{\partial^2 F}{\partial s_i \partial s_j}(t, S(t))\right\}_{i,j=1}^n. \end{aligned}$$

Note that $F_t$ and $F_i$ are scalar functions, whereas $F_{ss}$ is an $n \times n$ matrix-valued function.

We now form a portfolio based on $S_1, \ldots, S_n, B$ and $F$. For $S_1, \ldots, S_n$ and $F$ we use the notation $u_1, \ldots, u_n$ and $u_F$ for the corresponding portfolio weights, which gives us the weight $u_B$ for the money account as

$$u_B = 1 - \left(\sum_1^n u_i + u_F\right).$$

The dynamics of the value process for the corresponding self-financing portfolio are given by

$$dV = V\left[\sum_1^n u_i \frac{dS_i}{S_i} + u_F \frac{dF}{F} + u_B \frac{dB}{B}\right],$$

and substituting the expression for $u_B$ above, as well as inserting the dynamics of the processes involved, gives us

$$dV = V \cdot \left[\sum_1^n u_i(\alpha_i - r) + u_F(\alpha_F - r) + r\right]dt + V \cdot \left[\sum_1^n u_i\sigma_i + u_F\sigma_F\right]d\bar{W}.$$

We now try to choose the weights so that, first of all, the value process is locally risk free, i.e. it has no driving Wiener process. This means that we want to solve the equation

$$\sum_1^n u_i\sigma_i + u_F\sigma_F = 0.$$

Supposing for the moment that this can be done, we now have value dynamics of the form

$$dV = V \cdot \left[\sum_1^n u_i(\alpha_i - r) + u_F(\alpha_F - r) + r\right]dt,$$

and now we try to beat the market by choosing the weights such that we obtain a rate of return on the portfolio equaling $r + \beta$. Mathematically this means that we want to solve the equation

$$\sum_1^n u_i(\alpha_i - r) + u_F(\alpha_F - r) + r = r + \beta.$$

In order to see some structure, we now write these equations in matrix form as

$$\begin{bmatrix} \alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r \\ \sigma_1^* & \dots & \sigma_n^* & \sigma_F^* \end{bmatrix} \begin{bmatrix} u_S \\ u_F \end{bmatrix} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \tag{9.9}$$

(note that $\sigma_i^*$ is a column vector) where we have used the notation

$$u_S = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

Let us now take a closer look at the coefficient matrix in eqn (9.9). Denoting this matrix by $H$, we see that it is an $(n + 1) \times (n + 1)$ matrix, and we have two possibilities to consider, namely whether $H$ is invertible or not.

If $H$ is invertible, then the system (9.9) has a unique solution for every choice of β. In economic terms this means that we are able to form a self-financing portfolio with the dynamics

$$dV(t) = (r + \beta)V(t)dt,$$

which in turn means that we have constructed a "synthetic bank" with $r + \beta$ as its rate of interest. This will of course lead to arbitrage opportunities. We just solve the system for, say, $\beta = 0.10$, then we borrow a (large) amount of money from the bank and invest it in the portfolio. By this arrangement our net outlays at $t = 0$ are zero, our debt to the bank will increase at the rate $r$, whereas the value of our portfolio will increase at a rate which is 10% higher.

Since we assume absence of arbitrage we thus see that $H$ must be singular, and in order to see the implications of this we choose, for readability reasons, to study its transpose $H^*$, given by

$$H^* = \begin{bmatrix} \alpha_1 - r & \sigma_1 \\ \vdots & \vdots \\ \alpha_n - r & \sigma_n \\ \alpha_F - r & \sigma_F \end{bmatrix} \tag{9.10}$$

(recall that $\sigma_i$ is a row vector). We can write this somewhat more compactly by defining the $n$-dimensional column vector $1_n$ as

$$1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

With this notation we have

$$1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Since $H^*$ is singular this means that the columns are linearly dependent, and since the matrix $\sigma$ was assumed to be nonsingular we draw the conclusion that the first column of $H^*$ can be written as a linear combination of the other columns. Thus there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\alpha_i - r = \sum_{j=1}^{n} \sigma_{ij}\lambda_j, \; i = 1, \ldots, n, \tag{9.11}$$

$$\alpha_F - r = \sum_{j=1}^{n} \sigma_{Fj}\lambda_j, \tag{9.12}$$

where $\sigma_{Fj}$ denotes the $j$th component of the row vector $\sigma_F$.

There is an economic interpretation of the multipliers $\lambda_1, \ldots, \lambda_n$ as so called "market prices of risk" (cf. standard CAPM theory), and later on we will discuss this in some detail. For the moment the main logical point is that eqn (9.12) is the equation which will determine the derivative pricing function $F$. In order to use this equation we need, however, to get hold of the $\lambda$-vector, and as we shall see, this vector is determined by the system (9.11).

Writing (9.11) in vector form we see that the vector

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

is the solution of the $n \times n$ linear system

$$\alpha - r1_n = \sigma\lambda,$$

and since σ by assumption is nonsingular we see that λ is in fact uniquely determined as

$$\lambda = \sigma^{-1}[\alpha - r1_n]. \tag{9.13}$$

We now want to substitute this expression for λ into (9.12), but first we rewrite (9.12) in vector form as

$$\alpha_F - r = \sigma_F\lambda, \tag{9.14}$$

and from the definition of $\sigma_F$ in (9.8) we see that we may write $\sigma_F$ more compactly as

$$\sigma_F = \frac{1}{F} \cdot [S_1F_1, \ldots, S_nF_n]\sigma. \tag{9.15}$$

Inserting (9.13) and (9.15) into (9.14) and using $\sigma\sigma^{-1} = I$, we obtain the relation

$$\alpha_F - r = \frac{1}{F} \cdot [S_1F_1, \ldots, S_nF_n][\alpha - r1_n], \tag{9.16}$$

and finally we insert the expression for $\alpha_F$ from (9.7) into (9.16) to obtain, after some calculations,

$$F_t + \sum_{i=1}^{n} rS_iF_i + \frac{1}{2}tr\{\sigma^* D[S]F_{ss}D[S]\sigma\} - rF = 0. \tag{9.17}$$

Note that this equation is a stochastic equation. For ease of reading we have suppressed most of the arguments, but if we backtrack we will see that, for example, the term $rS_iF_i$ in the equation above is just shorthand for the expression

$$rS_i(t)\frac{\partial F}{\partial s_i}(t,S(t)).$$

Thus eqn (9.17) must hold, at each $t$, with probability 1 for every possible value of the price vector $S(t)$. Now, in our model for the price process it can be shown that the support (the set of possible values) for the vector $S(t)$ is the entire set $R^n_+$, and thus eqn (9.17) must also hold for each deterministic $t$ and $s$. By a standard argument we must also have $F(T,S(T)) = \Phi(S(T))$, so we have proved our main pricing result.

**Theorem 9.1** *Consider the contract* $\chi = \Phi(S(T))$. *In order to avoid arbitrage possibilities, the pricing function F(t,s) must solve the boundary value problem*

$$\begin{cases} F_t(t,s) + \sum_{i=1}^{n} rs_iF_i(t,s) + \frac{1}{2}tr\{\sigma^* D[S]F_{ss}D[S]\sigma\} - rF(t,s) = 0, \\ F(T,s) = \Phi(s). \end{cases} \tag{9.18}$$

**Remark 9.2.1** To be more explicit we recall that we can write the quadratic term above as

$$tr\{\sigma^* D[S] F_{ss} D[S]\sigma\} = \sum_{i,j=1}^{n} s_i s_j \frac{\partial^2 F}{\partial s_i \partial s_j}(t,s) C_{ij},$$

where

$$C_{ij} = [\sigma\sigma^*]_{ij}.$$

**Remark 9.2.2** As in the one-dimensional case we notice that the drift vector α, of the price process, does not appear in the pricing equation. Again we see that the only part of the underlying price process which influences the price of a financial derivative is the diffusion matrix σ. The reason for this phenomenon is the same as in the scalar case and we refer the reader to our earlier discussion on the subject.

**Remark 9.2.3** In the derivation of the result above we have assumed that the drift vector α and the volatility matrix σ in the price dynamics eqn (9.1) are constant. Going through the arguments it is, however, easily seen that we may allow the coefficients to be functions of current time and current stock price, i.e. they may be of the form

$$\begin{aligned} \alpha &= \alpha(t,S(t)), \\ \sigma &= \sigma(t,S(t)). \end{aligned}$$

If we assume that the volatility matrix σ (*t,s*) is invertible for each (*t,s*), then Theorem 9.1 will still hold, the only difference being that the term

$$tr\{\sigma^* D[S] F_{ss} D[S]\sigma\}$$

in the pricing equation is replaced by the term

$$tr\{\sigma^*(t,s) D[s] F_{ss}(t,s) D[s]\sigma(t,s)\}.$$

# 9.3 Risk Neutral Valuation

As in the scalar case there is a natural economic interpretation of Theorem 9.1 in terms of risk neutral valuation. From the Feynman–Kač representation theorem 4.8 we may immediately obtain a probabilistic formula for the solution to the pricing equation.

**Theorem 9.2** *The pricing function F(t,s) of Theorem 9.1 has the following representation.*

$$F(t,s) = e^{-r(T-t)} E_{t,s}^{Q}[\Phi(S(T))].$$

*Here the expectation is to be taken with respect to the martingale measure Q, defined by the fact that the Q-dynamics of the price process S are given by*

$$dS_i = rS_i dt + S_i \sigma_i dW, \; i = 1, \ldots, n.$$

*Adhering to the notational convention 6.4.1 the expression* $E^Q_{t,s}[\,]$ *indicates, as usual, that the expectation shall be taken under Q, given the initial condition S(t) = s. By the same convention, W is a Q-Wiener process.*

Again we see that the arbitrage free price of a derivative is given as the discounted expected value of the future cash flow, and again the main moral is that the expected value is **not** to be taken with respect to the objective probability measure *P*. Instead we must use the **martingale measure** *Q*. This martingale measure, or **risk adjusted measure** , is characterized by the following equivalent facts. The proof is left as an exercise to the reader.

**Proposition 9.3** *The martingale measure Q is characterized by any of the following equivalent conditions.*

1. *Under Q every price process* $\Pi(t)$, *be it underlying or derivative, has the risk neutral valuation property*

$$\Pi(t) = e^{-r(T-t)} E^Q_{t,s}[\Pi(T)].$$

2. *Under Q every price process* $\Pi(t)$, *be it underlying or derivative, has the short rate of interest as its local rate of return, i.e. the Q-dynamics are of the form*

$$d\Pi(t) = r\Pi(t)dt + \Pi(t)\sigma_{\Pi}(t)dW,$$

*where the volatility vector* $\sigma_{\Pi}$ *is the same under Q as under P.*

3. *Under Q every price process* $\Pi(t)$, *be it underlying or derivative, whas the property that the normalized price process*

$$\frac{\Pi(t)}{B(t)}$$

*is a martingale, i.e. it has a vanishing drift coefficient.*

As before we may summarize the moral as follows.

- When we compute arbitrage free prices of derivative assets, we can carry out the computations **as if** we live in a risk neutral world.
- This does **not** mean that we *de facto* live, or think that we live, in a risk neutral world.
- The formulas above hold regardless of the investor's preferences, and attitude towards risk, as long as he/she prefers more deterministic money to less.

# 9.4 Reducing the State Space

From Theorem 9.1 we see that in order to compute the price of a financial derivative based on *n* underlying assets, we have to solve a PDE with *n* state variables, and for a general case this has to be done by numerical methods. Sometimes it is, however, possible to reduce the dimension of the state space, and this can lead to a drastic simplification of the computational work, and in some cases even to analytical formulas. We will now present a theory which will allow us to obtain analytical pricing formulas for some nontrivial multidimensional claims which quite often occur in practice. The theory presented here is based on an analysis of the pricing PDE, but there also exists a corresponding probabilistic theory. See Chapter 19 below.

Let us assume that we have the model

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t)\sum_{j=1}^{n} \sigma_{ij} d\bar{W}_j(t), \; i = 1, \ldots, n. \tag{9.19}$$

We consider a $T$-claim of the form $\chi = \Phi(S(T))$, and the crucial assumptions are the following.

**Assumption 9.4.1**

- *For the rest of the section we assume that the contract function* $\Phi$ *is homogeneous of degree 1, i.e. that*

$$\Phi(t \cdot s) = t \cdot \Phi(s),$$

  *for all* $t > 0$ *and for all* $s \in R^n$.
- *The volatility matrix* $\sigma$ *is constant.*

For a homogeneous $\Phi$ we see that, by choosing $t = s_n^{-1}$, we have the relation

$$\Phi(s_1, \ldots, s_n) = s_n \Phi\left(\frac{s_1}{s_n}, \ldots, \frac{s_{n-1}}{s_n}, 1\right).$$

This naturally gives us the Idea that perhaps also the corresponding pricing function $F$ has the same homogeneity property, so we try the **ansatz**

$$F(t, s_1, \ldots, s_n) = s_n G\left(t, \frac{s_1}{s_n}, \ldots, \frac{s_{n-1}}{s_n}\right) \tag{9.20}$$

where $G$ is some function

$$G: R_+ \times R^{n-1} \rightarrow R.$$

First of all, if there is a solution to the pricing PDE of the form (9.20), then it has to satisfy the boundary Condition

$$F(T, s) = \Phi(s), \; \forall s,$$

and translated into $G$ this gives the boundary condition

$$G(T, z) = \Psi(z), \; \forall z, \tag{9.21}$$

where the function $\Psi : R^{n-1} \rightarrow R$ is defined by

$$\Psi(z_1, \ldots, z_{n-1}) = \Phi(z_1, \ldots, z_{n-1}, 1). \tag{9.22}$$

The main problem is now to see whether there is a function $G$ such that the ansatz (9.20) satisfies the pricing PDE. We therefore compute the various partial derivatives of $F$ in terms of $G$, and substitute the result into the PDE. After some tedious calculations we have the following relations, where for brevity $z$ denotes the vector $z = \left(\frac{s_1}{s_n}, \ldots, \frac{s_{n-1}}{s_n}\right)$. Subscripts denotes partial derivatives.

$$\begin{aligned}
F_t(t,s) &= s_n G_t(t,z), \\
F_i(t,s) &= G_i(t,z), \ i = 1, \ldots, n-1, \\
F_n(t,s) &= G(t,z) - \sum_{j=1}^{n-1} \frac{s_j}{s_n} G_j(t,z), \\
F_{ij}(t,s) &= \frac{1}{s_n} G_{ij}(t,z), \ i,j = 1, \ldots, n-1, \\
F_{in}(t,s) &= F_{ni}(t,s) = -\sum_{j=1}^{n-1} \frac{s_j}{s_n^2} G_{ij}(t,z), \ i = 1, \ldots, n-1, \\
F_{nn} &= -\sum_{i,j=1}^{n-1} \frac{s_i s_j}{s_n^3} G_{ij}(t,z).
\end{aligned}$$

As in Remark 9.2.1 we write the pricing PDE as

$$F_t(t,s) + \sum_{i=1}^{n} r s_i F_i(t,s) + \frac{1}{2} \sum_{i,j=1}^{n-1} s_i s_j F_{ij}(t,s) C_{ij} - rF(t,s) = 0. \tag{9.23}$$

Substituting the expressions above for the partial derivatives of $F$ into eqn (9.23) may look fairly forbidding, but in fact there will be many cancelations, and we end up with the following PDE for $G$.

$$G_t(t,z) + \frac{1}{2} \sum_{i,j=1}^{n-1} z_i z_j G_{ij}(t,z) D_{ij} = 0, \tag{9.24}$$

where

$$D_{ij} = C_{ij} + C_{nn} + C_{in} + C_{nj}. \tag{9.25}$$

We have thus proved the following result.

**Proposition 9.4** *Assume that the contract function $\Phi$ is homogeneous of degree 1, and that the volatility matrix $\sigma$ is constant. Then the pricing function $F$ is given by*

$$F(t,s) = s_n G\left(t, \frac{s_1}{s_n}, \ldots, \frac{s_{n-1}}{s_n}\right),$$

*where $G(t, z_1, \ldots, z_{n-1})$ solves the boundary value problem*

$$
\begin{aligned}
G_t(t,z) + \frac{1}{2} \sum_{i,j=1}^{n-1} z_i z_j G_{ij}(t,z) D_{ij} &= 0, \\
G(T,z) &= \Psi(z).
\end{aligned}
\tag{9.26}
$$

*Here the matrix D is defined in (9.25), where C is given by Remark 9.2.1. The function Ψ is defined in (9.22).*

The point of this result is that, instead of having to solve a PDE with $n$ state variables (and one time variable), we have reduced the problem to that of solving a PDE with only $n-1$ state variables.

**Remark 9.4.1** The crucial assumption in Proposition 9.4 is of course the homogeneity of the contract function, but one may wonder where we used the assumption of a constant volatility matrix. The point is that, with a state dependent $\sigma$, the PDE (9.24) is no longer a PDE in only the $n-1$ state variables $z_1, \ldots, z_{n-1}$, because the matrix $D$ now depends on the entire price vector $s_1, \ldots, s_n$.

It is interesting to note that the PDE satisfied by $G$ is of parabolic type. By inspection we see that it can in fact be interpreted as the pricing PDE, in a world with zero interest rate, for a claim of the form $Y = \Psi(Z(T))$, where the $n-1$ underlying price processes satisfy an SDE of the form

$$dZ(t) = D[Z]\mu(t)dt + D[Z]\tilde{\sigma} d\tilde{W}(t),$$

where the drift vector $\mu$ is as usual of no importance, the process $W$ is an $(n-1)$-dimensional standard $P$-Wiener process, and the covariance matrix satisfies

$$\tilde{\sigma}\tilde{\sigma}^* = D.$$

**Example 9.5** We illustrate the technique of state space reduction by a simple example of some practical importance. We consider a model with two underlying price processes $S_1$ and $S_2$, satisfying (under $P$) the following system of SDEs.

$$
\begin{aligned}
dS_1 &= S_1\alpha_1 dt + S_1\sigma_1 d\bar{W}_1, \\
dS_2 &= S_2\alpha_2 dt + S_2\sigma_2 d\bar{W}_2.
\end{aligned}
$$

As usual $W_1$ and $W_2$ are independent, so in this model the price processes are also independent, but this is just for notational simplicity.

The claim to be studied is an **exchange option** , which gives the holder the right, but not the obligation, to exchange one $S_2$ share for one $S_1$ share at time $T$. Formally this means that the claim is defined by $\chi = \max[S_1(T) - S_2(T), 0]$, and we see that we have indeed a contract function which is homogeneous of degree 1. A straightforward application of Theorem 9.1 would give us the PDE

$$\begin{aligned} F_t + rs_1F_1 + rs_2F_2 + \frac{1}{2}\sigma_1^2F_{11} + \frac{1}{2}\sigma_2^2F_{22} - rF &= 0, \\ F(T,s_1,s_2) &= \max[s_1 - s_2, 0]. \end{aligned}$$

Using Proposition 9.4 we can instead write

$$F(t,s_1,s_2) = s_2 \cdot G\left(t, \frac{s_1}{s_2}\right),$$

where $G(t,z)$ satisfies

$$\begin{aligned} G_t(t,z) + \frac{1}{2}z^2G_{zz}(t,z)(\sigma_1^2 + \sigma_2^2) &= 0, \\ G(T,z) &= \max[z - 1, 0]. \end{aligned}$$

We see that this is the pricing equation, in a world with short rate $r = 0$, for a European call option with strike price $K = 1$ written on a single stock with volatility $\sqrt{\sigma_1^2 + \sigma_2{}^2}$. Thus $G$ can be obtained directly from the Black–Scholes formula, and after some minor calculations we get

$$F(t,s_1,s_2) = s_1N[d_1(t,z)] - s_2N[d_2(t,z)],$$

where $z = \frac{s_1}{s_2}$. $N$ is as usual the cumulative distribution function for the $N$ [0,1] distribution and

$$\begin{aligned} d_1(t,z) &= \frac{1}{\sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}}\left\{ \ln z + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t) \right\}, \\ d_2(t,z) &= d_1(t,z) - \sqrt{(\sigma_1^2 + \sigma_2^2)(T-t)}. \end{aligned}$$

## 9.5 Hedging

When we introduced the model

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t)\sigma_i d\bar{W}(t) \tag{9.27}$$

for the price vector process, one reason for the assumption of an invertible volatility matrix was that we wanted a complete market, and the goal of this section is to show that our model is in fact complete. From Chapter 7 we recall the following definition.

**Definition 9.6** *We say that a T-claim* $\chi$ *can be* **replicated** , *alternatively that it is* **reachable** *or* **hedgeable** , *if there exists a self financing portfolio h such that*

$$V^h(T) = \chi, \ P - a.s. \tag{9.28}$$

*In this case we say that h is a* **hedge** *against* $\chi$, *alternatively a* **replicating portfolio** *for* $\chi$. *If every contingent claim is reachable we say that the market is* **complete** .

Since we lack the necessary probabilistic machinery, we will not be able to show that we can hedge **every** contingent *T*-claim $\chi$. As in the scalar case, we will "only" be able to prove completeness for simple claims, i.e. we will prove that every claim of the form

$$\chi = \Phi(S(T)),$$

can be replicated.

Let us thus fix a date of delivery *T*, and a claim $\chi = \Phi(S(T))$. We denote portfolio weights by $u^0, u^1, \ldots, u^n$, where $u^i$ is the weight on asset *i* for $i = 1, \ldots, n$, whereas $u^0$ is the weight on the risk free asset. From Lemma 5.5 it is clear that if we can find a process *V*, and weight processes $u^0, u^1, \ldots, u^n$ such that

$$V(T) = \Phi(S(T)), P - a.s. \tag{9.29}$$

$$dV = V\left\{u^0 \frac{dB}{B} + \sum_{i=1}^{n} u^i \frac{dS_i}{S_i}\right\}, \tag{9.30}$$

$$\sum_{i=0}^{n} u^i = 1, \tag{9.31}$$

then $u^0, u^1, \ldots, u^n$ is a replicating portfolio for $\chi$, and *V* is the corresponding value process. For future use we note that eqns (9.30)–(9.31) can be written as

$$dV = V \cdot \left\{r\left(1 - \sum_{i=1}^{n} u^i\right) + \sum_{i=1}^{n} u^i \alpha_i\right\}dt + V \cdot \sum_{i=1}^{n} u^i \sigma_i d\bar{W}. \tag{9.32}$$

The structure of the proof is that we will make an educated guess about the nature of the value process *V*, so we will make an ansatz. Given this ansatz we then compute the stochastic differential *dV*, and at last, comparing the expression thus obtained to (9.32), we identify the portfolio weights by inspection.

We now go on to produce a natural candidate for the role as value process, and to this end we recall from Section 7.1 that, for a hedgeable claim, we should have the relation $\Pi(t, \chi) = V(t)$.

Thus the obvious candidate as the value process for the replicating portfolio (if it exists at all) is the price process $\Pi(t, \chi)$ for the claim. On the other hand we have already computed the price process as

$$\Pi(t, \chi) = F(t, S(t)),$$

where *F* solves the pricing PDE of Theorem 9.1.

Let us thus **define** *F* as the solution to the pricing PDE, and then **define** the process *V* by $V(t) = F(t, S(t))$. Our task is to show that *V* in fact is the value process of a replicating portfolio, i.e. to show that the relation (9.29) is satisfied, and that the dynamics for *V* can be written in the form (9.32). Equation (9.29) is obviously (why?) satisfied, and using the identity (by definition) $F = V$, we obtain from (9.6)

$$dV = F\alpha_F dt + F\sigma_F d\overline{W}.$$

So, from (9.7)–(9.8)

$$\begin{aligned} dV &= V \cdot \frac{F_t + \Sigma_1^n \alpha_i S_i F_i + \frac{1}{2} tr(\sigma^* D[S] F_{ss} D[S] \sigma)}{F} dt \\ &+ V \cdot \sum_1^n \frac{S_i F_i}{F} \sigma_i d\overline{W}. \end{aligned} \tag{9.33}$$

Comparing the diffusion part of (9.33) to that of (9.32) we see that the natural candidates as $u^1, \ldots, u^n$ are given by

$$u^i(t) = \frac{S_i F_i}{F}, \; i = 1, \ldots, n. \tag{9.34}$$

Substituting these expressions into (9.33) gives us

$$dV = V \cdot \left\{ \sum_1^n u^i \alpha_i + \frac{F_t + \frac{1}{2} tr(\sigma^* D[S] F_{ss} D[S] \sigma)}{F} \right\} dt + V \cdot \sum_1^n u^i \sigma_i d\overline{W},$$

and, using the fact that $F$ satisfies the pricing PDE, we obtain

$$dV = V \cdot \left\{ \sum_1^n u^i \alpha_i + \frac{-\Sigma_1^n r S_i F_i + rF}{F} \right\} dt + V \cdot \sum_1^n u^i \sigma_i d\overline{W}.$$

Again using the definition (9.34), this reduces to

$$dV = V \cdot \left\{ \left(1 - \sum_1^n u^i\right) r + \sum_1^n u^i \alpha_i \right\} dt + V \cdot \sum_1^n u^i \sigma_i d\overline{W}, \tag{9.35}$$

which is exactly eqn (9.32). We have thus proved the second part of the following theorem.

**Theorem 9.7** *Assume that the volatility matrix* $\sigma$ *is invertible. Then the following hold.*

- *The market is complete, i.e. every claim can be replicated.*
- *For a T-claim* $\mathcal{X}$ *of the form* $\mathcal{X} = \Phi(S(T))$, *the weights in the replicating portfolio are given by*

$$\begin{aligned} u^i(t) &= \frac{S_i(t) F_i(t, S(t))}{F(t, S(t))}, \; i = 1, \ldots, n, \\ u^0(t) &= 1 - \sum_{i=1}^n u^i(t), \end{aligned}$$

*where* $F$ *by definition is the solution of the pricing PDE in Theorem 9.1.*

We have only proved that every **simple** contingent claim can be replicated, so we have not proved the first item above in full generality. Using more advanced probabilistic tools (the martingale representation theorem for Wiener filtrations) it can in fact be proved that every (sufficiently integrable) claim can be replicated, but this is beyond the scope of the present text.

## 9.6 Exercises

**Exercise 9.1** Prove Proposition 9.3.

**Exercise 9.2** Check all calculations in the derivation of the PDE in Proposition 9.4.

**Exercise 9.3** Consider again the exchange option in Example 9.5. Now assume that $W_1$ and $W_2$ are no longer independent, but that the local correlation is given by $dW_1 \cdot dW_2 = \rho \, dt$. (We still assume that both Wiener processes have unit variance parameter, i.e. that $d\bar{W}_1^2 = d\bar{W}_2^2 = dt$.) How will this affect the Black–Scholes-type formula given in the example?

**Exercise 9.4** Consider the stock price model in Example 9.5. The $T$-contract $\chi$ to be priced is defined by

$$\chi = \max\left[aS_1(T), bS_2(T)\right],$$

where $a$ and $b$ are given positive numbers. Thus, up to the scaling factors $a$ and $b$, we obtain the maximum of the two stock prices at time $T$. Use Proposition 9.4 and the Black–Scholes formula in order to derive a pricing formula for this contract. See Johnson (1987).

**Hint:** You may find the following formula (for $x > 0$) useful.

$$\max\left[x, 1\right] = 1 + \max\left[x - 1, 0\right].$$

**Exercise 9.5** Use the ideas in Section 9.4 to analyze the pricing PDE for a claim of the form $\chi = \Phi(S(T))$ where we now assume that $\Phi$ is homogeneous of degree $\beta$, i.e.

$$\Phi(t \cdot s) = t^{\beta} \Phi(s), \quad \forall t > 0.$$

# 10 Incomplete Markets

## 10.1 Introduction

In this chapter we will investigate some aspects of derivative pricing in incomplete markets. We know from the meta-theorem that markets generically are incomplete when there are more random sources than there are traded assets, and this can occur in an infinite number of ways, so there is no "canonical" way of writing down a model of an incomplete market. We will confine ourselves to study a particular type of incomplete market, namely a "factor model", i.e. a market where there are some nontraded underlying objects. Before we go on to the formal description of the models let us briefly recall what we may expect in an incomplete model.

- Since, by Assumption, the market is incomplete we will not be able to hedge a generic contingent claim.
- In particular there will not be a unique price for a generic derivative.

## 10.2 A Scalar Nonpriced Underlying Asset

We will start by studying the simplest possible incomplete market, namely a market where the only randomness comes from a scalar stochastic process which is **not** the price of a traded asset. We will then discuss the problems which arise when we want to price derivatives which are written in terms of the underlying object. The model is as follows.

**Assumption 10.2.1** *The only objects which are a priori given are the following.*

- *An empirically observable stochastic process $X$, which is* **not** *assumed to be the price process of a traded asset, with P-dynamics given by*

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))d\overline{W}(t). \tag{10.1}$$

*HereWis a standard scalar P-Wiener process.*

- *A risk free asset (money account) with the dynamics*

$$dB(t) = rB(t)dt, \tag{10.2}$$

*where r as usual is the deterministic short rate of interest.*

We now consider a given contingent claim, written in terms of the process $X$. More specifically we define the $T$-claim $\mathcal{Y}$ by

$$\mathcal{Y} = \Phi(X(T)), \tag{10.3}$$

where $\Phi$ is some given deterministic function, and our main problem is that of studying the price process $\Pi(t; \mathcal{Y})$ for this claim.

In order to give some substance to the discussion, and to understand the difference between the present setting and that of the previous chapters, let us consider a specific concrete, interpretation of the model. We may, for example, interpret the process $X$ as the temperature at some specific point on the earth, say the end of the Palace Pier in Brighton. Thus $X(t)$ is the temperature (in centigrade) at time $t$ at the Palace Pier. Suppose now that you want to go to Brighton for a holiday, but that you fear that it will be unpleasantly cold at the particular time $T$ when you visit Brighton. Then it may be wise to buy "holiday insurance", i.e. a contract which pays you a certain amount of money if the weather is unpleasant at a prespecified time in a prespecified place. If the monetary unit is the pound sterling, the contract function $\Phi$ above may have the form

$$\Phi(x) = \begin{cases} 100, & \text{if } x \le 20, \\ 0, & \text{if } x > 20. \end{cases}$$

In other words, if the temperature at time $T$ is below 20°C (degrees centigrade) you will obtain 100 pounds from the insurance company, whereas you will get nothing if the temperature exceeds 20°C.

The problem is now that of finding a "reasonable" price for the contract above, and as usual we interpret the word "reasonable" in the sense that there should be no arbitrage possibilities if we are allowed to trade the contract. This last sentence contains a hidden Assumption which we now formalize.

**Assumption 10.2.2** *There is a liquid market for every contingent claim.*

If we compare this model with the standard Black–Scholes model, we see many similarities. In both cases we have the money account $B$, and in both cases we have an a priori given underlying process. For the Black–Scholes model the underlying process is the stock price $S$, whereas we now have the underlying process $X$, and in both models the claim to be priced is a deterministic function $\Phi$ of the underlying process, evaluated at time $T$.

In view of these similarities it is now natural to assume that the results from the Black–Scholes analysis will carry over to the present case, i.e. we are (perhaps) led to believe that the price process for the claim $\mathcal{Y}$ is uniquely determined by the $P$-dynamics of the underlying process $X$. It is, however, very important to understand that this is, most emphatically, **not** the case, and the reasons are as follows.

1. If we consider the a priori given market, which only consists of the money account $B$, we see that the number $R$ of random sources in this case equals

one (one driving Wiener process), while the number $M$ of traded assets (always excluding the money account $B$) equals zero. From the meta-theorem it now follows that the market is incomplete. The incompleteness can also be seen from the obvious fact that in the a priori given market there are no interesting ways of forming self-financing portfolios. The only strategy that is allowed is to invest all our money in the bank, and then we can only sit down and passively watch our money grow at the rate $r$. In particular we have no possibility to replicate any interesting derivative of the form $\Phi\ (X(T))$. We thus conclude that, since we cannot replicate our claim, we cannot expect to obtain a unique arbitrage free price process.

2. One natural strategy to follow in order to obtain a unique price for the claim $X$ is of course to imitate the scheme for the Black–Scholes model. We would assume that the price process $\Pi\ (t, Y)$ is of the form $\Pi\ (t, Y) = F(t, X(t))$, and then we would form a portfolio based on the derivative $F$ and the underlying $X$. Choosing the portfolio weights such that the portfolio has no driving Wiener process would give us a riskless asset, the rate of return on which would have to equal the short rate $r$, and this last equality would finally have the form of a PDE for the pricing function $F$. This approach is, however, completely nonsensical, since in the present setting the process $X$ is (by Assumption) **not the price of a traded asset** , and thus it is meaningless to talk about a "portfolio based on $X$". In our concrete interpretation this is eminently clear. Obviously you can buy any number of insurance contracts and put them in your portfolio, but it is also obvious that you cannot meaningfully add, for example, 15°C to that portfolio.

We can summarize the situation as follows.

- The price of a particular derivative will **not** be completely determined by the specification (10.1) of the $X$-dynamics and the requirement that the market $[B(t),\ \Pi\ (t, Y)]$ is free of arbitrage.
- The reason for this fact is that arbitrage pricing is always a case of pricing a derivative **in terms of** the price of some underlying assets. In our market we do not have sufficiently many underlying assets.

Thus we will not obtain a unique price of a particular derivative. This fact does not mean, however, that prices of various derivatives can take any form whatsoever. From the discussion above we see that the reason for the incompleteness is that we do not have enough underlying assets, so if we adjoin one more asset to the market, without introducing any new Wiener processes, then we expect the market to be complete. This Idea can be expressed in the following ways.

Idea 10.2.1

- We **cannot** say anything about the price of any **particular** derivative.
- The requirement of an arbitrage free derivative market implies that **prices of different derivatives** (i.e. claims with different contract functions or

different times of expiration) will have to satisfy certain **internal consistency relations** in order to avoid arbitrage possibilities on the bond market. In terms of our concrete interpretation this means that even if we are unable to produce a unique price for a fixed weather insurance contract, say the "20°C contract"

$$\Phi(x) = \begin{cases} 100, & \text{if } x \le 20, \\ 0, & \text{if } x > 20, \end{cases}$$

for the fixed date $T$, there must be internal consistency requirements between the price of this contract and the price of the following "25°C contract"

$$\Gamma(x) = \begin{cases} 100, & \text{if } x \le 25, \\ 0, & \text{if } x > 25, \end{cases}$$

with some expiration date $T'$ (where of course we may have $T=T'$).

- In particular, if we take the price of **one** particular "benchmark" derivative as a priori given, then the prices of all other derivatives will be uniquely determined by the price of the benchmark. This fact is in complete agreement with the meta-theorem, since in an a priori given market consisting of one benchmark derivative plus the risk free asset we will have $R=M=1$, thus guaranteeing completeness. In our concrete interpretation we expect that the prices of all insurance contracts should be determined by the price of any fixed benchmark contract. If, for example, we choose the 20°C contract above as our benchmark, and take its price as given, then we expect the 25°C contract to be priced uniquely in terms of the benchmark price.

To put these ideas into action we now take as given two fixed $T$-claims, $Y$ and $Z$, of the form

$$\begin{aligned} Y &= \Phi(X(T)), \\ Z &= \Gamma(X(T)), \end{aligned}$$

where $\Phi$ and $\Gamma$ are given deterministic real valued functions. The project is to find out how the prices of these two derivatives must be related to each other in order to avoid arbitrage possibilities on the derivative market. As above we assume that the contracts are traded on a frictionless market, and as in the Black–Scholes analysis we make an Assumption about the structure of the price processes.

**Assumption 10.2.3** *We assume that*

- *There is a liquid, frictionless market for each of the contingent claims $Y$ and $Y$.*
- *The market prices of the claims are of the form*

$$\begin{aligned} \Pi\ (t, Y) &= F(t, X(t)), \\ \Pi\ (t, Z) &= G(t, X(t)), \end{aligned}$$

*where $F$ and $G$ are smooth real valued functions.*

We now proceed exactly as in the Black–Scholes case. We form a portfolio based on $F$ and $G$, and choose the weights so as to make the portfolio locally riskless. The rate of return of this riskless portfolio has to equal the short rate of interest, and this relation will give us some kind of equation, which we then have to analyze in detail.

From the Assumption above, the Itô formula, and the $X$-dynamics we obtain the following price dynamics for the price processes $F(t,X(t))$ and $G(t,X(t))$.

$$dF = \alpha_F F dt + \sigma_F F d\overline{W}, \tag{10.4}$$

$$dG = \alpha_G G dt + \sigma_G G d\overline{W}. \tag{10.5}$$

Here the processes $\alpha_F$ and $\sigma_F$ are given by

$$\begin{aligned} \alpha_F &= \frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}}{F}, \\ \sigma_F &= \frac{\sigma F_x}{F}, \end{aligned}$$

and correspondingly for $\alpha_G$ and $\sigma_G$. As usual we have suppressed most arguments, so in more detail we have, for example,

$$\mu F_x = \mu(t,X(t))\frac{\partial F}{\partial x}(t,X(t)).$$

We now form a self-financing portfolio based on $F$ and $G$, with portfolio weights denoted by $u_F$ and $u_G$ respectively. According to (5.13), the portfolio dynamics are given by

$$dV = V\left\{u_F \cdot \frac{dF}{F} + u_G \cdot \frac{dG}{G}\right\},$$

and using the expressions above we get

$$dV = V\left\{u_F \cdot \alpha_F + u_G \cdot \alpha_G\right\} dt + V\left\{u_F \cdot \sigma_F + u_G \cdot \sigma_G\right\} d\overline{W}. \tag{10.6}$$

In order to make this portfolio locally riskless we must choose $u_F$ and $u_G$ such that $u_F \cdot \sigma_F + u_G \cdot \sigma_G = 0$, and we must also remember that they must add to unity. Thus we define $u_F$ and $u_G$ as the solution to the following system of equations.

$$\begin{cases} u_F + u_G &= 1, \\ u_F \cdot \sigma_F + u_G \cdot \sigma_G &= 0. \end{cases}$$

The solution to this system is given by

$$\begin{cases} u_F &= \dfrac{-\sigma_G}{\sigma_F - \sigma_G}, \\ u_G &= \dfrac{\sigma_F}{\sigma_F - \sigma_G}, \end{cases}$$

and inserting this into the portfolio dynamics equation (10.6) gives us

$$dV = V \cdot \left\{ \frac{\alpha_G \cdot \sigma_F - \alpha_F \cdot \sigma_G}{\sigma_F - \sigma_G} \right\} dt.$$

We have thus created a locally riskless asset, so using Proposition 6.6, absence of arbitrage must imply the equation

$$\frac{\alpha_G \cdot \sigma_F - \alpha_F \cdot \sigma_G}{\sigma_F - \sigma_G} = r.$$

After some reshuffling we can rewrite this equation as

$$\frac{\alpha_F - r}{\sigma_F} = \frac{\alpha_G - r}{\sigma_G}.$$

The important fact to notice about this equation is that the left hand side does not depend on the choice of $G$, while the right hand side does not depend upon the choice of $F$. The common quotient will thus depend neither on the choice of $F$ nor on the choice of $G$, and we have proved the following central Result.

**Proposition 10.1** *Assume that the market for derivatives is free of arbitrage. Then there exists a universal process* $\lambda(t)$ *such that, with probability 1, and for all t, we have*

$$\frac{\alpha_F(t) - r}{\sigma_F(t)} = \lambda(t), \tag{10.7}$$

*regardless of the specific choice of the derivative F.*

There is a natural economic interpretation of this Result, and of the process $\lambda$. In eqn (10.7) the numerator is given by $\alpha_F - r$, and from (10.4) we recognize $\alpha_F$ as the local mean rate of return on the derivative $F$. The numerator $\alpha_F - r$ is thus the local mean excess return on the derivative $F$ over the riskless rate of return $r$, i.e. the risk premium of $F$. In the denominator we find the volatility $\sigma_F$ of the $F$ process, so we see that $\lambda$ has the dimension "risk premium per unit of volatility". This is a concept well known from CAPM theory, so $\lambda$ is commonly called "the market price of risk". Proposition 10.1 can now be formulated in the following, slightly more flashy, form.

- In a no arbitrage market all derivatives will, regardless of the specific choice of contract function, have the same market price of risk.

We can obtain more explicit information from eqn (10.7) by substituting our earlier formulas for $\alpha_F$ and $\sigma_F$ into it. After some algebraic manipulations we then end up with the following PDE.

$$F_t + \{\mu - \lambda\sigma\} F_x + \frac{1}{2}\sigma^2 F_{xx} - rF = 0.$$

This is really shorthand notation for an equation which must hold with probability 1, for each $t$, when all terms are evaluated at the point $(t, X(t))$. Assuming for the moment that the support of $X$ is the entire real line, we can then draw the conclusion that the equation must also hold identically when we evaluate it at an arbitrary deterministic point $(t,x)$. Furthermore it is clear that we must have the boundary Condition

$$F(T,x) = \Phi(x), \quad \forall x \in R,$$

so we finally end up with the following Result.

**Proposition 10.2 (Pricing equation)** *Assuming absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\Phi(X(T))$ solves the following boundary value problem.*

$$F_t(t,x) + AF(t,x) - rF(t,x) = 0, \ (t,x) \in (0,T) \times R, \tag{10.8}$$

$$F(T,x) = \Phi(x), \ x \in R, \tag{10.9}$$

*where*

$$AF(t,x) = \{\mu(t,x) - \lambda(t,x)\sigma(t,x)\} F_x(t,x) + \frac{1}{2}\sigma^2(t,x) F_{xx}(t,x).$$

At first glance this Result may seem to contradict the moral presented above. We have stressed earlier the fact that, because of the incompleteness of the market, there will be **no** unique arbitrage free price for a particular derivative. In Proposition 10.2, on the other hand, we seem to have arrived at a PDE which, when solved, will give us precisely the unique pricing function for any simple claim. The solution to this conundrum is that the pricing equation above is indeed very nice, but in order to solve it we have to know the short rate of interest $r$, as well as the functions $\mu(t,x)$, $\sigma(t,x)$, $\Phi(x)$ and $\lambda(t,x)$. Of these, only $r$, $\mu(t,x)$, $\sigma(t,x)$ and $\Phi(x)$ are specified exogenously. The market price of risk $\lambda$, on the contrary, is **not** specified within the model. We can now make Idea 10.2.1 above more precise.

Firstly we see that, even though we cannot determine a unique price for a particular derivative, prices of different derivatives must satisfy internal consistency requirements. This requirement is formulated precisely in Proposition 10.1, which says that all derivatives must have the same market price of risk. Thus, if we consider two different derivative assets with price processes $F$ and $G$, these may have completely different local mean rates of return ($\alpha_F$ and $\alpha_G$), and they may also have completely different volatilities ($\sigma_F$ and $\sigma_G$). The consistency relation which has to be satisfied in order to avoid arbitrage between the derivatives is that, at all times, the quotient

$$\frac{\alpha_F - r}{\sigma_F}$$

must equal

$$\frac{\alpha_G - r}{\sigma_G}.$$

Secondly, let us assume that we take the price process of one particular derivative as given. To be concrete let us fix the "benchmark" claim $\Gamma(X(T))$ above and assume that the pricing function $G(t,x)$, for $\Gamma(X(T))$, is specified exogenously. Then we can compute the market price of risk by the formula

$$\lambda(t,x) = \frac{\alpha_G(t,x) - r}{\sigma_G(t,x)}. \tag{10.10}$$

Let us then consider an arbitrary pricing function, say the function $F$ for the claim $\Phi\,(X(T))$. Since the market price of risk is the same for all derivatives we may now take the expression for $\lambda$ obtained from (10.10) and insert this into the pricing equation (10.8)–(10.9) for the function $F$. Now everything in this equation is well specified, so we can (in principle) solve it, in order to obtain $F$. Thus we see that the price $F$ of an arbitrary claim is indeed uniquely determined by the price $G$ of any exogenously specified benchmark claim.

We can obtain more information from the pricing equation by applying the Feynman–Kač representation. The Result can be read off immediately and is as follows.

**Proposition 10.3 (Risk neutral valuation)** *Assuming absence of arbitrage, the pricing function $F(t,x)$ of the T-claim $\Phi\,(X(T))$ is given by the formula*

$$F(t,x) = e^{-r(T-t)} E^{Q}_{t,x}\left[\Phi(X(T))\right], \tag{10.11}$$

*where the dynamics of $X$ under the martingale measure $Q$ are given by*

$$dX(t) = \{\mu(t,X(t)) - \lambda(t,X(t))\sigma(t,X(t))\}\,dt + \sigma(t,X(t))dW(t).$$

*Here $W$ is a $Q$-Wiener process, and the subscripts $t,x$ indicate as usual that $X(t) = x$.*

Again we have an "explicit" risk neutral valuation formula for the pricing function. The arbitrage free price of the claim is given as the discounted value of the mathematical expectation of the future claim. As before the expectation is, however, not to be taken under the objective measure $P$, but under the martingale ("risk adjusted") measure $Q$. Note that there is a one-to-one correspondence between the martingale measure and the market price of risk. Thus, choosing a particular $\lambda$ is equivalent to choosing a particular $Q$. The characterization of $Q$ is just the same as in Proposition 9.3. The proof is left to the reader.

**Proposition 10.4** *The martingale measure $Q$ is characterized by any of the following equivalent facts.*

- *The local mean rate of return of any derivative price process $\Pi\,(t)$ equals the short rate of interest, i.e. the $\Pi\,(t)$-dynamics have the following structural form under $Q$*

  $$d\Pi(t) = r\Pi(t)dt + \sigma_{\Pi}\Pi(t)(t)dW,$$

  *where $W$ is a $Q$-Wiener process, and $\sigma_{\Pi}$ is the same under $Q$ as under $P$.*
- *With $\Pi$ as above, the process $\Pi\,(t)/B(t)$ is a $Q$-martingale, i.e. it has a zero drift term.*

As for the pricing equation above, we have to know $\lambda$ in order to compute the expected value in the risk neutral valuation formula. Thus, in the present context, and in contrast to the Black–Scholes case, the martingale measure $Q$ is not (generically) determined within the model.

It is instructive to see how the standard Black–Scholes model can be interpreted within the present model. Let us therefore assume, in contrast to our earlier assumptions, that $X$ **is** in fact the price of a stock in a Black–Scholes model. This of course means that the $P$-dynamics of $X$ are given by

$$dX = \alpha G dt + \sigma G d\overline{W},$$

for some constants $\alpha$ and $\sigma$. Thus, in terms of the notation in (10.1), $\mu(t,x) = x \cdot \alpha$ and $\sigma(t,x) = x \cdot \sigma$. Our project is still that of finding the arbitrage free price for the claim $\Phi(X(T))$, and to this end we follow the arguments given above, which of course are still valid, and try to find a suitable benchmark claim $\Gamma$. In the present setting we can make a particularly clever choice of $\Gamma(X(T))$, and in fact we define the claim by

$$\Gamma(X(T)) = X(T).$$

This is the claim which at time $T$ gives us exactly one share of the underlying stock, and the point is that the price process $G(t,X(t))$ for this claim is particularly simple. We have in fact $G(t,X(t)) = X(t)$, the reason being that the claim can be trivially replicated using a buy-and-hold strategy which consists of one unit of the stock itself. Note that for this argument to hold we use critically our new Assumption that $X$ really is the price of a traded asset. Using the identity $dG = dX$ we now trivially obtain the $G$-dynamics as

$$dG = \alpha X dt + \sigma X d\overline{W}.$$

Thus, in terms of our earlier discussion we have $\alpha_G(t,x) = \alpha$ and $\sigma_G(t,x) = \sigma$. We can now determine the market price of risk as

$$\lambda(t,x) = \frac{\alpha_G - r}{\sigma_G} = \frac{\alpha - r}{\sigma},$$

and the point to note is that, because of our tradability Assumption, $\lambda$ is now determined **within** the model. We now go on to the pricing PDE, into which we substitute $\lambda$ given as above. The critical part, $\mathcal{A}F$, of the PDE now becomes

$$\begin{aligned} \mathcal{A}F(t,x) &= \{\mu(t,x) - \lambda(t,x)\sigma(t,x)\} F_x(t,x) + \frac{1}{2}\sigma^2(t,x)F_{xx}(t,x) \\ &= \left\{\alpha x - \frac{\alpha - r}{\sigma} \cdot \sigma x\right\} F_x + \frac{1}{2}x^2\sigma^2 F_{xx} \\ &= rxF_x + \frac{1}{2}x^2\sigma^2 F_{xx}, \end{aligned}$$

so we end up (as expected) with the Black–Scholes equation

$$\begin{cases} F_t + rxF_x + \frac{1}{2}x^2\sigma^2F_{xx} - rF &= 0, \\ F(T,x) &= \Phi(x), \end{cases}$$

and in the risk neutral valuation formula we will get the old Black–Scholes $Q$-dynamics

$$dX = rX\,dt + \sigma X\,dW.$$

## 10.3 The Multidimensional Case

We will now go on to study pricing in a model with more than one non-priced underlying asset. The model is as follows.

**Assumption 10.3.1** *The only objects which are a priori given are the following.*

- *An empirically observable $k$-dimensional stochastic process*

$$X = (X_1, \ldots, X_k),$$

*which is* **not** *assumed to be the price process of a traded asset, with P-dynamics given by*

$$dX_i(t) = \mu_i(t,X(t))dt + \delta_i(t,X(t))d\bar{W}(t),\ i = 1, \ldots, k, \tag{10.12}$$

*where $\bar{W} = (W_1, \ldots, W_n)^*$ is a standard n-dimensional P-Wiener process.*
- A risk free asset (money account) with the dynamics

$$dB(t) = rB(t)dt. \tag{10.13}$$

The object is to find an arbitrage free price process for a $T$-claim $\mathcal{Y}$ of the form

$$\mathcal{Y} = \Phi(X(T)).$$

Drawing on our experiences from the previous two sections we expect the following.

- We cannot say anything precise about the price process of any particular contingent claim.
- Different claims will, however, have to satisfy certain internal consistency requirements in order to avoid arbitrage on the derivative market.
- More precisely, since we now have $n$ sources of randomness we can (generically) specify the price processes of $n$ different "benchmark" claims. The price processes of all other claims will then be uniquely determined by the prices of the benchmarks.

As usual we assume that there is a liquid market for all contingent claims, and in order to use the ideas above we fix exactly $n$ claims $Y_1, \ldots, Y_n$ of the form

$$Y_i = \Phi_i(t, X(T)), \; i = 1, \ldots, n.$$

We furthermore assume that the price processes, $[\Pi^1(t), \ldots, \Pi^n(t)]$, for these claims exist, and that they are of the form

$$\Pi^i(t) = F^i(t, X(t)), \; i = 1, \ldots, n.$$

**Remark 10.3.1** Note that we use superscripts to distinguish between the various pricing functions. This is done in order to allow us to use subscripts to denote partial derivatives.

These claims are our "benchmarks", and we now study the claim

$$Y = \Phi(X(T))$$

above, the price of which is assumed to have the form

$$\Pi(t) = F(t, X(t)).$$

Using the same notational conventions as in Section 9.2, the price dynamics of the various derivatives are given by

$$\begin{aligned} dF &= \alpha_F F dt + \sigma_F F d\overline{W}, \\ dF^i &= \alpha_i F^i dt + \sigma_i F^i d\overline{W}, \quad i = 1, \ldots, n, \end{aligned}$$

where

$$\alpha_i = \frac{F^i_t + \sum_{j=1}^k F^i_j \mu_j + \frac{1}{2} tr\{\delta^* F^i_{xx} \delta\}}{F^i}, \tag{10.14}$$

$$\sigma_i = \frac{\sum_{j=1}^k F^i_j \delta_j}{F^i}, \tag{10.15}$$

$$\alpha_F = \frac{F_t + \sum_{j=1}^k F_j \mu_j + \frac{1}{2} tr\{\delta^* F_{xx} \delta\}}{F}, \tag{10.16}$$

$$\sigma_F = \frac{\sum_{j=1}^k F_j \delta_j}{F}. \tag{10.17}$$

We can now follow the reasoning of Section 9.2 word for word. To recapitulate, we form a self-financing portfolio based on the price processes $[B, F, F^1, \ldots, F^n]$. We then choose the weights so as to make the portfolio locally riskless, and finally we try to "beat the bank" by forcing the portfolio to have a higher rate of return than the short rate $r$.

The conclusion will once again be that, in order to have an arbitrage free market, the matrix

$$H^* = \begin{bmatrix} \alpha_1 - r & \sigma_1 \\ \vdots & \vdots \\ \alpha_n - r & \sigma_n \\ \alpha_F - r & \sigma_F \end{bmatrix}$$

(10.18)

must be singular (recall that $\sigma_i$ is a row vector). Assuming that the $n \times n$ matrix

$$\sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix}$$

is invertible (with probability 1 for each $t$) we thus deduce the existence of multiplier processes $\lambda_1\,(t,X(t)),\ldots,\lambda_n(t,X(t))$ such that

$$\alpha_i - r = \sum_{j=1}^{n} \sigma_{ij}\lambda_j,\ i = 1,\ldots,n,$$

and

$$\alpha_F - r = \sum_{j=1}^{n} \sigma_{Fj}\lambda_j.$$

(10.19)

Since the claim $\Phi\,(X(T))$ was chosen arbitrarily we see that the risk premium, $\alpha_F - r$, of any asset can be written as a linear combination of the volatility components, $\sigma_{Fj}$, of the asset, the important point being that the multipliers are the same for all assets. The vector process $\lambda(t,X(t))$ is (see Sections 9.2 and 10.2) known as the **market price (vector) of risk** (cf. CAPM), and we see that the individual component $\lambda_j$ has the dimension "risk premium per unit of $j$-type volatility".

Using the notational conventions of Section 9.2, the $\lambda$-vector is determined by the equation

$$\alpha - r1_n = \sigma\lambda,$$

i.e.

$$\lambda = \sigma^{-1}[\alpha - r1_n].$$

(10.20)

This is precisely eqn (9.13), but in the present setting the volatility matrix $\sigma$ and the vector of returns $\alpha$ are no longer given a priori, so $\lambda$ is not determined within the model. We can summarize this as follows.

- If the derivatives are traded in a no arbitrage market, then there will exist a market price of risk vector which is the same for all assets.
- The market price of risk is not determined within the a priori specified model.

- If we exogenously specify the prices of any $n$ assets such that the corresponding volatility matrix process σ is nonsingular with probability 1 for all $t$, then the market price of risk will be uniquely determined by this specification and by eqn (10.20). Thus all derivatives will be priced **in terms of** the benchmark prices.

Note that we can choose the (smooth) benchmark pricing functions above in any way whatsoever, subject only to the following conditions.

- The boundary conditions $F(T,x) = \Phi_i(x), i = 1, \ldots, n$ are satisfied.
- The volatility matrix $\sigma(t,x)$ is invertible for all $(t,x)$.

The first Condition is obvious. The second is the mathematical formulation of the requirement that the family of benchmark derivatives is rich enough to span the entire space of derivatives.

We can easily obtain a PDE for derivative pricing, by writing out eqn (10.19) in detail. Using (10.16)–(10.17) we obtain, after some simplification, the following Result.

**Proposition 10.5 (Pricing equation)** it If the market is arbitrage free, then the arbitrage free price process $\Pi(t;\Phi)$ *is given by* $\Pi(t;\Phi) = F(t,X(t))$, *where* $F$ *solves the boundary value problem*

$$\begin{cases} F_t + \sum_{i=1}^{k}\left(\mu_i - \sum_{1}^{n}\delta_{ij}\lambda_j\right)F_i + \frac{1}{2}tr\{\delta^* F_{xx}\delta\} - rF = 0, \\ F(T,x) = \Phi(x), \end{cases}$$

*where* $\lambda_1, \ldots, \lambda_n$ *are universal in the sense that they do not depend on the specific choice of derivative.*

A standard application of the Feynman–Kač technique gives us a risk neutral valuation formula.

**Proposition 10.6 (Risk neutral valuation)** *If the market is arbitrage free, then there exists a martingale measure* $Q$*, such that the pricing function* $F$ *in the proposition above can be represented as*

$$F(t,x) = e^{-r(T-t)}E_{t,x}^{Q}[\Phi(X(T))], \tag{10.21}$$

*where the* $Q$*-dynamics of the process* $X$ *are given by*

$$dX^i = \{\mu_i - \delta_i\lambda\}\,dt + \delta_i dW, \; i = 1, \ldots, k.$$

*where* $\lambda_1, \ldots, \lambda_n$ *are universal.*

We have the usual characterization of the martingale measure $Q$.

**Proposition 10.7** *The martingale measure* $Q$ *has the following properties.*

1. *Under $Q$ every price process $\Pi(t)$, be it underlying or derivative, has the risk neutral valuation property*

$$\Pi(t) = e^{-r(T-t)}E^{Q}_{t,s}[\Pi(T)].$$

2. *Under $Q$ every price process $\Pi(t)$, be it underlying or derivative, has the short rate of interest as its local rate of return, i.e. the $Q$-dynamics are of the form*

$$d\Pi(t) = r\Pi(t)dt + \Pi(t)\sigma_{\Pi}(t)dW,$$

*where the volatility vector $\sigma_{\Pi}$ is the same under $Q$ as under $P$.*

3. *Under $Q$ every price process $\Pi(t)$, be it underlying or derivative, has the property that the normalized price process*

$$\frac{\Pi(t)}{B(t)}$$

*is a martingale, i.e. it has a vanishing drift coefficient.*

**Remark 10.3.2** The model formulation above also includes the case when some, or all, of the $X$-components, say $X_1, \ldots, X_m$, are traded assets (e.g. exogenously given stock prices), whereas the remaining components (if any) are nontraded state variables. As a special case we see that if $m = k = n$, then we are back in the case of a complete market treated in Chapter 9.

## 10.4 A Stochastic Rate of Interest

In the theory derived above we have assumed a constant short rate of interest $r$. Let us now assume that we have exactly the same situation as in Section 10.3 with the following difference.

**Assumption 10.4.1** *The short rate of interest is assumed to be a deterministic function of the factors, i.e.*

$$r(t) = r(X(t)). \tag{10.22}$$

Here we have used a slightly sloppy notation: the $r$ on the left hand side denotes a stochastic process, whereas the $r$ appearing on the right hand side denotes a deterministic function. We have thus assumed that the factor vector $X$ completely determines the short rate, and as a special case we can of course have one of the factors equal to the short rate itself.

For this model, we can now go through all our earlier arguments once again, and we will easily obtain the following version of Proposition 10.5.

**Proposition 10.8 (Pricing equation)** *The arbitrage free price process $\Pi(t;\Phi)$ is given by $\Pi(t;\Phi) = F(t,X(t))$, where $F$ solves the boundary value problem*

$$\begin{cases} F_t + \sum_{i=1}^{k}\left(\mu_i - \sum_{1}^{n}\delta_{ij}\lambda_j\right)F_i + \frac{1}{2}tr\{\delta^* F_{xx}\delta\} - r(x)F & = 0, \\ F(T,x) & = \Phi(x), \end{cases}$$

*where $\lambda_1, \ldots, \lambda_n$ are the same for all derivatives.*

Again a standard application of the Feynman–Kač technique gives us a risk neutral valuation formula.

**Proposition 10.9 (Risk neutral valuation)** *There exists a martingale measure Q, such that the pricing function F in the proposition above can be represented as*

$$F(t,x) = E_{t,x}^{Q}\left[e^{-\int_t^T r(X(u))du} \cdot \Phi(X(T))\right], \tag{10.23}$$

*where the Q-dynamics of the process X are given by*

$$dX^i = \{\mu_i - \delta_i \lambda\} dt + \delta_i dW, \; i = 1, \ldots, k,$$

*and* $\lambda_1, \ldots, \lambda_n$ *are the same for all derivatives.*

## 10.5 Summing Up

We may now sum up our experiences from the preceding sections.

Result 10.5.1

- In an arbitrage free market, regardless of whether the market is complete or incomplete, there will exist a market price of risk process, $\lambda(t)$, which is common to all assets in the market. More precisely, let $\Pi(t)$ be any price process in the market, with *P*-dynamics

$$d\Pi(t) = \Pi(t)\alpha_{\Pi}(t)dt + \Pi(t)\sigma_{\Pi}(t)d\bar{W}(t).$$

  *Then the following holds, for all t, and P-a.s.*

$$\alpha_{\Pi}(t) - r = \sigma_{\Pi}(t)\lambda(t).$$

- In a complete market the price of any derivative will be **uniquely** determined by the requirement of absence of arbitrage. In hedging terms this means that the price is unique because the derivative can equally well be replaced by its replicating portfolio. Phrased in terms of pricing PDEs and risk neutral valuation formulas, the price is unique because a complete market has the property that the martingale measure *Q*, or equivalently the market price of risk $\lambda$, is uniquely determined within the model.
- In an incomplete market the requirement of no arbitrage is no longer sufficient to determine a unique price for the derivative. We have several possible martingale measures, and several market prices of risk. The reason that there are several possible martingale measures simply means that there are several different price systems for the derivatives, all of which are consistent with absence of arbitrage.

Schematically speaking the price of a derivative is thus determined by two major factors.

- We require that the derivative should be priced in such a way so as to not introduce arbitrage possibilities into the market. This requirement is reflected by the fact that all derivatives must be priced by formula (10.11) where the same $Q$ is used for all derivatives, or equivalently by the pricing PDE (10.8)–(10.9), where the same $\lambda$ is used for all derivatives.
- In an incomplete market the price is also partly determined, in a nontrivial way, by aggregate supply and demand on the market. Supply and demand for a specific derivative are in turn determined by the aggregate risk aversion on the market, as well as by liquidity considerations and other factors. All these aspects are aggregated into the particular martingale measure used by the market.

When dealing with derivative pricing in an incomplete market we thus have to fix a specific martingale measure $Q$, or equivalently a $\lambda$, and the question arises as to how this is to be done.

**Question:**

**Who chooses the martingale measure?**

From the discussions above the answer should by now be fairly clear.

**Answer:**

**The market!**

The main implication of this message is that, within our framework, it is **not** the job of the theorist to determine the "correct" market price of risk. The market price of risk is determined on the market, by the agents in the market, and in particular this means that if we assume a particular structure of the market price of risk, then we have implicitly made an Assumption about the preferences on the market. To take a simple example, suppose that in our computations we assume that $\lambda = 0$. This means in fact that we have assumed that the market is risk neutral (why?).

From this it immediately follows that if we have a concrete model, and we want to obtain information about the prevailing market price of risk, then we must go to the concrete market and get that information using empirical methods. It would of course be nice if we were able to go to the market and ask the question "What is today's market price of risk?", or alternatively "Which martingale measure are you using?", but for obvious reasons this cannot be done in real life. The information that **can** be obtained from the market is **price** data, so a natural Idea is to obtain implicit information about the market price of risk using the existing prices on the market. This method, sometimes called "calibrating the model to market data", "backing out the parameters", or "computing the implied parameter values", can schematically be described as follows.

Let us take a concrete model as given, say the one defined by eqn (10.1). We assume that we know the exact form of μ and σ. Our problem is that of pricing a fixed claim Φ (*X*(*T*)), and in order to do this we need to know the market price of risk λ (*t*,*x*), and we have to consider two possible scenarios for the derivative market, which we are trying to model.

The first case occurs if there is no existing market for any derivative product with *X* as its underlying object, i.e. no weather insurance contracts for Brighton are traded. Then we are stuck. Since the market price of risk is determined by the market, then if there is no market, there is no market price of risk.

The second case occurs if some contracts are already traded. Let us thus assume that there exists a market for the claims $\Phi_i$ (*X*(*T*)), *i* = 1,. . . , *n*. Let us furthermore assume that we want to choose our market price of risk from a parameterized family of functions, i.e. we assume a priori that λ is of the form

$$\lambda = \lambda(t,x;\beta),\ \beta \in R^k.$$

We have thus a priori specified the functional form of λ but we do not know which parameter vector β we should use. We then carry out the following scheme. We are standing at time *t* = 0.

- Compute the theoretical pricing functions $F^i$ (*t*,*x*) for the claims $\Phi_1, \cdots, \Phi_n$. This is done by solving the pricing PDE for each contract, and the Result will of course depend on the parameter β, so the pricing functions will be of the form

$$F^i = F^i(t,x;\beta),\ i = 1,\ldots,\ n.$$

- In particular, by observing today's value of the underlying process, say *X*(0) = $x_0$, we can compute today's **theoretical prices** of the contracts as

$$\Pi^i(0;\beta) = F^i(0,x_0;\beta).$$

- We now go to the concrete market and observe the actually traded prices for the contracts, thus obtaining the **observed prices**

$$\Pi^{i*}(0),$$

  where the superscript * indicates an observed value.
- We now choose the “implied” parameter vector $\beta^*$ in such a way that the theoretical prices are “as close as possible” to the observed prices, i.e. such that

$$\Pi^{i*}(0) \approx \Pi^i(0;\beta^*),\ i = 1,\ldots,\ n.$$

  One way, out of many, of formalizing this step is to determine $\beta^*$ by solving the least squares minimization problem

$$\min_{\beta \in R^k} \left[ \sum_{i=1}^{n} \{\Pi^i(0;\beta) - \Pi^{i*}(0)\}^2 \right].$$

In the theory of interest rates, this procedure is widely used under the name of "the inversion of the yield curve" and in that context we will study it in some detail.

As we have repeatedly stressed, the problem of determining the market price of risk is not a theoretical one, but an empirical one. It should, however, be pointed out that the truth of this statement depends on our particular framework, where we have not specified individual preferences at all. An alternative framework is that of a general equilibrium model, where we have specified the utility functions of a number of individuals, and the production functions of a number of firms. In such a model all prices, derivative or underlying, as well as the market price of risk, will be determined endogenously within the model, so we no longer have to go to the market to find λ. The price we have to pay for a general equilibrium approach is that we must then specify the individual preferences, which in turn is essentially equivalent to a specification of the market price of risk.

## 10.6 Exercises

**Exercise 10.1** Consider a claim $\Phi(X(T))$ with pricing function $F(t,x)$. Prove Proposition 10.4, i.e. prove that $dF$ under $Q$ has the form

$$dF = rFdt + \{\dots\}dW,$$

where $W$ is a $Q$-Wiener process.

**Hint:** Use Itô's formula on $F$, using the $Q$-dynamics of $X$. Then use the fact that $F$ satisfies the pricing PDE.

**Exercise 10.2** Convince yourself, either in the scalar or in the multidimensional case, that the market price of risk process λ really is of the form

$$\lambda = \lambda(t, X(t)).$$

**Exercise 10.3** Prove Proposition 10.7.

**Exercise 10.4** Consider the scalar model in Section 10.2 and a fixed claim $\Gamma(X(T))$. Take as given a pricing function $G(t,x)$, for this claim, satisfying the boundary Condition $G(T,x)=\Gamma(x)$, and assume that the corresponding volatility function $\sigma_G(t,x)$ is nonzero. We now expect the market $[B,G]$ to be complete. Show that this is indeed the case, i.e. show that every simple claim of the form $\Phi(X(T))$ can be replicated by a portfolio based on $B$ and $G$.

**Exercise 10.5** Consider the multidimensional model in Section 10.3 and a fixed family of claims $\Phi_i(X(T))$, $i = 1, \dots, n$. Take as given a family of pricing functions $F^i(t,x)$, $i = 1, \dots, n$, for these claims, satisfying the boundary Condition $F^i(T,x) = \Phi_i(x)$, $i = 1, \dots, n$, and assume that the corresponding volatility matrix $\sigma(t,x)$ is nonzero. Show that the market $[B, F^1, \dots, F^n]$ is complete, i.e. show that every simple claim of the form $\Phi(X(T))$ can be replicated by a portfolio based on $[B, F^1, \dots, F^n]$.

**Exercise 10.6** Prove the propositions in Section 10.4.

# 11 Dividends

The object of the present chapter is to study pricing problems for contingent claims which are written on dividend paying underlying assets. In real life the vast majority of all traded options are written on stocks having at least one dividend left before the date of expiration of the option. Thus the study of dividends is important from a practical point of view. Furthermore, it turns out that the theory developed in this chapter will be of use in the study of currency derivatives and we will also need it in connection with futures contracts.

## 11.1 Discrete Dividends

### 11.1.1 Price Dynamics and Dividend Structure

We consider an underlying asset ("the stock") with price process $S$, over a fixed time interval $[0,T]$. We take as given a number of deterministic points in time, $T_1, \ldots, T_n$, where

$$0 < T_n < T_{n-1} < \ldots < T_2 < T_1 < T.$$

The interpretation is that at these points in time dividends are paid out to the holder of the stock. We now go on to construct a model for the stock price process as well as for the dividend structure, and we assume that, under the objective probability measure $P$, the stock price has the following dynamics, **between dividends** .

$$dS = \alpha S dt + \sigma S d\bar{W}. \tag{11.1}$$

To be quite precise we assume that the $S$-process satisfies the SDE above on each half open interval of the form $[T_{i+1}, T_i)$, $i = 1, \ldots, n - 1$, as well as on the intervals $[0,T_n)$ and $[T_1,T]$.

The first conceptual issue that we have to deal with concerns the interpretation of $S_t$ as "the price of the stock at time $t$", and the problem to be handled is the following: do we regard $S_t$ as the price immediately **after** , or immediately **before** , the payment of a dividend? From a logical point of view we can choose any interpretation—nothing is affected in real terms, but our choice of interpretation will affect the notation below. We will in fact choose the first interpretation, i.e. we view the stock price as the price **ex dividend** . If we think of the model on an infinitesimal time scale we have the interpretation that, if $t$ is a dividend point, dividends are paid out, not at the time point $t$, but rather at $t - dt$ or, if you will, at $t-$.

Next we go on to model the size of the dividends.

**Assumption 11.1.1** *We assume as given a deterministic continuous function* δ [*s*]

$$\delta : R \to R$$

*The dividend* δ, *at a dividend time t, is assumed to have the form*

$$\delta = \delta[S_{t-}].$$

Note that the assumption above guarantees that the size of the dividend δ at a dividend time $t$ is already determined at $t-$.

Our next problem concerns the behavior of the stock price at a dividend point $t$, and an easy arbitrage argument (see the exercises) gives us the following result.

**Proposition 11.1 (Jump condition)** *In order to avoid arbitrage possibilities the following jump condition must hold at every dividend point t:*

$$S_t = S_{t-} - \delta[S_{t-}]. \tag{11.2}$$

The stock price structure can now be summarized as follows.

- Between dividend points the stock price process satisfies the SDE

$$dS = \alpha S dt + \sigma S d\bar{W}.$$

- Immediately before a dividend time $t$, i.e. at $t- = t - dt$, we observe the stock price $S_{t-}$.
- Given the stock price above, the size of the dividend is determined as δ [$S_{t-}$].
- "Between" $t - dt$ and $t$ the dividend is paid out.
- At time $t$ the stock price has a jump, determined by

$$S_t = S_{t-} - \delta[S_{t-}].$$

## 11.1.2 Pricing Contingent Claims

As usual we consider a fixed contingent $T$-claim of the form

$$\chi = \Phi(S_T),$$

where Φ is some given deterministic function, and our immediate problem is to find the arbitrage free price process $\Pi(t;\chi)$ for the claim $\chi$. We will solve this problem by a recursive procedure where, starting at $T$ and then working backwards in time, we will compute $\Pi(t;\chi)$ for each intra-dividend interval separately.

$T_1 \le t \le T$:

We start by computing $\Pi(t;\chi)$ for $t \in [T_1, T]$. Since our interpretation of the stock price is ex dividend, this means that we are actually facing a problem without dividends over this interval. Thus, for $T_1 \leq t \leq T$, we have $\Pi(t;\chi) = F(t,S_t)$ where $F$ solves the usual Black–Scholes equation

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

In particular, the pricing function at $T_1$ is given by $F(T_1,s)$.

$T_2 \leq t < T_1$:

Now we go on to compute the price of the claim for $T_2 \leq t \leq T_1$. We start by computing the pricing function $F$ at the time immediately before $T_1$, i.e. at $t = T_1-$. Suppose therefore that we are holding one unit of the contingent claim, and let us assume that the price at time $T_1-$ is $S_{T_1-} = s$. This is the price cum dividend, and in the next infinitesimal interval the following will happen.

- The dividend $\delta[s]$ will be paid out to the shareholders.
- At time $T_1$ the stock price will have dropped to $s - \delta[s]$.
- We are now standing at time $T_1$, holding a contract which is worth $F(T_1, s - \delta[s])$. The value of $F$ at $T_1$ has, however, already been computed in the previous step, so we have the **jump condition**

$$F(T_1-,s) = F(T_1,s-\delta[s]). \tag{11.3}$$

It now remains to compute $F$ for $T_2 \leq t < T_1$, but this turns out to be quite easy. We are holding a contingent claim on an underlying asset which over the interval $[T_2, T_1)$ is not paying dividends. Thus the standard Black–Scholes argument applies, which means that $F$ has to solve the usual Black–Scholes equation

$$\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} - rF = 0,$$

over this interval. The boundary value is now given by the jump condition (11.3) above. Another way of putting this is to say that, over the *half open* interval $[T_2, T_1)$, we have $F(t,s) = F^1(t,s)$ where $F^1$ solves the following boundary value problem over the *closed* interval $[T_2, T_1]$.

$$\begin{cases} \frac{\partial F^1}{\partial t} + rs\frac{\partial F^1}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F^1}{\partial s^2} - rF^1 &= 0, \\ F^1(T,s) &= F(T_1,s-\delta[s]). \end{cases}$$

Thus we have determined $F$ on $[T_2, T_1)$.

$T_3 \leq t < T_2$:

Now the story repeats itself. At $T_2$ we will again have the jump condition

$$F(T_2-,s) = F(T_2,s-\delta[s]),$$

and over the half open interval $[T_3, T_2)$ $F$ has to satisfy the Black–Scholes equation.

We may summarize our results as follows.

**Proposition 11.2 (Pricing equation)** *The pricing function F(t,s) is determined by the following recursive procedure.*

- *On the interval* $[T_1, T]$ *F solves the boundary value problem*

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases} \tag{11.4}$$

- *At each dividend point* $T_i$, *F has to satisfy the jump condition*

$$F(T_i -, s) = F(T_i, s - \delta[s]). \tag{11.5}$$

- *On every half open interval of the form* $[T_{i+1}, T_i)$, $i = 1,2,\ldots n-1$, *as well as on the interval* $[0, T_n)$, *F solves the Black–Scholes equation*

$$\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2} - rF = 0. \tag{11.6}$$

Another way of formulating this result in order to stress the recursive nature of the procedure is as follows.

**Proposition 11.3**

- *On the interval* $[T_1, T]$ *we have* $F(t,s) = F^0(t,s)$, *where* $F^0$ *solves the boundary value problem*

$$\begin{cases} \frac{\partial F^0}{\partial t} + rs\frac{\partial F^0}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F^0}{\partial s^2} - rF^0 &= 0, \\ F^0(T,s) &= \Phi(s). \end{cases} \tag{11.7}$$

- *On each half open interval* $[T_{i+1}, T_i)$ *we have* $F(t,s) = F^i(t,s)$ for $i = 1,2,\ldots$, *where* $F^i$, *over the closed interval* $[T_{i+1}, T_i]$, *solves the boundary value problem*

$$\begin{cases} \frac{\partial F^i}{\partial t} + rs\frac{\partial F^i}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F^i}{\partial s^2} - rF^i &= 0, \\ F^i(T_i,s) &= F^{i-1}(T_i, s - \delta[s]) \end{cases} \tag{11.8}$$

Throughout the entire section we have assumed the standard Black–Scholes price dynamics (11.1) between dividends. It is easy to see that we also have the following more general result.

**Proposition 11.4** *Assume that the stock price dynamics between dividends are of the form*

$$dS_t = S_t\alpha(t,S_t)dt + S_t\sigma(t,S_t)d\bar{W}_t. \tag{11.9}$$

*Assume furthermore that the dividend structure as before is given by*

$$\delta = \delta[S_{t-}].$$

*Then the results of Propositions 11.2 and 11.3 still hold, provided that the constant* σ *is replaced by the function* $\sigma(t,s)$ *in the PDEs.*

We now turn to the possibility of obtaining a probabilistic "risk neutral valuation" formula for the contingent claim above, and as in the PDE approach this is done in a recursive manner.

For $T_1 \leq t \leq T$ the situation is simple. Since we have no dividend points left we may use the old risk neutral valuation formula to obtain

$$F^0(t,s) = e^{-r(T-t)} E^Q_{t,s}[\Phi(S_T)]. \tag{11.10}$$

Here the $Q$-dynamics of the stock price are given by

$$dS = rSdt + \sigma SdW,$$

and we have used the notation $F^0$ to emphasize that in this interval there are zero dividend points left. Note that the $Q$-dynamics of $S$ above are only defined for the interval $[T_1, T]$.

For $T_2 \leq t < T$ the situation is slightly more complicated. From Proposition 11.3 we know that the pricing function, which on this interval is denoted by $F^1$, solves the PDE

$$\begin{cases} \dfrac{\partial F^1}{\partial t} + rs\dfrac{\partial F^1}{\partial s} + \dfrac{1}{2}s^2\sigma^2\dfrac{\partial^2 F^1}{\partial s^2} - rF^1 & = 0, \\ F^1(T_1,s) & = F^0(T_1,s - \delta[s]) \end{cases}$$

We may now apply the Feynman–Kač Theorem 4.6 to obtain the stochastic representation

$$F^1(t,s) = e^{-r(T_1-t)} E_{t,s}[F_0(T_1, X_{T_1} - \delta[X_{T_1}])] \tag{11.11}$$

where the process $X$, which at this point only acts as a computational dummy, is defined by

$$dX = rXdt + \sigma XdW. \tag{11.12}$$

Notice that the dummy process $X$ is defined by (11.12) over the entire **closed** interval $[T_2, T_1]$ (whereas the pricing function $F^1$ is the relevant pricing function only over the half open interval $[T_2, T_1)$). This implies that $X$ has continuous trajectories over the closed interval $[T_2, T_1]$, so in particular we see that $X_{T_1^-} = X_{T_1}$. We may thus rewrite (11.11) as

$$F^1(t,s) = e^{-r(T_1-t)} E_{t,s}\Big[F^0(T_1, X_{T_1-} - \delta[X_{T_1-}])\Big] \tag{11.13}$$

still with the $X$-dynamics (11.12) on the closed interval $[T_2, T_1]$.

Let us now define the $Q$-dynamics for $S$ over the closed interval $[T_2, T_1]$ by writing

$$dS = rSdt + \sigma SdW,$$

for the half open interval $[T_2, T_1)$ and adding the jump condition

$$S_{T_1} = S_{T_1-} - \delta[S_{T_1-}].$$

In this notation we can now write (11.13) as

$$F^1(t,s) = e^{-r(T_1-t)} E^Q_{t,s}\left[F^0(T_1, S_{T_1})\right],$$

and, plugging in our old expression for $F^0$, we obtain

$$F^1(t,s) = e^{-r(T_1-t)} E^Q_{t,s}\left[e^{-r(T-T_1)} E^Q_{T_1,S(T_1)}\left[\Phi(S_T)\right]\right].$$

Taking the discount factor out of the expectation, and using standard rules for iterated condition al expectations, this formula can be reduced to

$$F^1(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right].$$

We may now iterate this procedure for each intra-dividend interval to obtain the following risk neutral valuation result, which we formulate in its more general version.

**Proposition 11.5 (Risk neutral valuation)** *Consider a T-claim of the form* $\Phi(S_T)$ *as above. Assume that the price dynamics between dividends are given by*

$$dS_t = \alpha(t,S_t)S_t\, dt + \sigma(t,S_t)S_t\, d\bar{W},$$

*and that the dividend size at a dividend point t is given by*

$$\delta = \delta[S_{t-}].$$

*Then the arbitrage free pricing function F(t,s) has the representation*

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right],$$

*where the Q-dynamics of S between dividends are given by* (11.14)

$$dS_t = rS_t\, dt + \sigma(t,S_t)S_t\, dW,$$

*with the jump condition* (11.15)

$$S_t = S_{t-} - \delta[S_{t-}]$$

*at each dividend point, i.e. at* $t = T_1, T_2, \ldots, T_n$. (11.16)

We end this section by specializing to the case when we have the standard Black–Scholes dynamics

$$dS = \alpha S dt + \sigma S d\bar{W}$$

between dividends, and the dividend structure has the particularly simple form (11.17)

$$\delta[s] = s\delta,$$

where δ on the right hand side denotes a positive constant. (11.18)

As usual we consider the $T$-claim $\Phi(S_T)$ and, in order to emphasize the role of the parameter δ, we let $F_\delta(t,s)$ denote the pricing function for the claim Φ. In particular we observe that $F_0$ is our standard pricing function for Φ in a model with no dividends at all. Using the risk neutral valuation formula above, it is not hard to prove the following result.

**Proposition 11.6** *Assume that the P-dynamics of the stock price and the divi-dend structure are given by* (*11.17*)–(*11.18*). *Then the following relation holds.*

$$F_\delta(t,s) = F_0(t,(1-\delta)^n \cdot s),$$

(11.19)

*where n is the number of dividend points in the interval (t,T].*

**Proof** See the exercises.

The point of this result is of course that in the simple setting of (11.17)–(11.18) we may use our "old" formulas for no dividend models in order to price contingent claims in the presence of dividends. In particular we may use the standard Black–Scholes formula for European call options in order to price call options on a dividend paying stock. Note, however, that in order to obtain these nice results we must assume both (11.17) and (11.18).

# 11.2 Continuous Dividends

In this section we consider the case when dividends are paid out continuously in time. As usual $S_t$ denotes the price of the stock at time $t$, and by $D(t)$ we denote the cumulative dividends over the interval $[0,t]$. Put in differential form this means that over the infinitesimal interval $(t,t+dt]$ the holder of the stock receives the amount $dD(t) = D(t+dt) - D(t)$.

## 11.2.1 Continuous Dividend Yield

We start by analyzing the simplest case of continuous dividends, which is when we have a **continuous dividend yield** .

**Assumption 11.2.1** *The price dynamics, under the objective probability measure, are given by*

$$dS_t = S_t \cdot \alpha(S_t)\ dt + S_t \cdot \sigma(S_t)\ d\overline{W}_t.$$

(11.20)

*The dividend structure is assumed to be of the form*

$$dD(t) = S_t \cdot \delta[S_t]\ dt,$$

(11.21)

*where* δ *is a continuous deterministic function.*

The most common special case is of course when the functions α and σ above are deterministic constants, and when the function δ is a deterministic constant. We note that, since we have no discrete dividends, we do not have to worry about the interpretation of the stock price as being ex dividend or cum dividend.

The problem to be solved is again that of determining the arbitrage free price for a $T$-claim of the form $\Phi\ (S_T)$. This turns out to be quite easy, and we can in fact follow the strategy of Chapter 6. More precisely we recall the following scheme.

1. Assume that the pricing function is of the form $F(t,S_t)$.
2. Consider α, σ, Φ, $F$, δ and $r$ as exogenously given.

3. Use the general results from Section 5.2 to describe the dynamics of the value of a hypothetical self-financed portfolio based on the derivative instrument and the underlying stock.
4. Form a self-financed portfolio whose value process $V$ has a stochastic differential without any driving Wiener process, i.e. it is of the form

$$dV(t) = V(t)k(t)\ dt.$$

5. Since we have assumed absence of arbitrage we must have $k = r$.
6. The condition $k = r$ will in fact have the form of a partial differential equation with $F$ as the unknown function. In order for the market to be efficient $F$ must thus solve this PDE.
7. The equation has a unique solution, thus giving us the unique pricing formula for the derivative, which is consistent with absence of arbitrage.

We now carry out this scheme and, since the calculations are very close to those in Chapter 6, we will be rather brief.

Denoting the relative weights of the portfolio invested in the stock and in the derivative by $u_S$ and $u_F$ respectively we obtain (see Section 5.3) the value process dynamics as

$$dV = V \cdot \left\{ u_S \frac{dG_S}{S} + u_F \frac{dF}{F} \right\},$$

where the gain differential $dG_S$ for the stock is given by

$$dG_S = dS + dD,$$

i.e.

$$dG_S = S(\alpha + \delta)dt + \sigma S d\overline{W}.$$

From the Itô formula we have the usual expression for the derivative dynamics

$$dF = \alpha_F F dt + \sigma_F F d\overline{W},$$

where

$$\begin{aligned} \alpha_F &= \frac{1}{F}\left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\}, \\ \sigma_F &= \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}. \end{aligned}$$

Collecting terms in the value equation gives us

$$dV = V \cdot \{u_S(\alpha + \delta) + u_F \alpha_F\}\, dt + V \cdot \{u_S \sigma + u_F \sigma_F\}\, d\overline{W},$$

and we now determine the portfolio weights in order to obtain a value process without a driving Wiener process, i.e. we define $u_S$ and $u_F$ as the solution to the system

$$u_S \sigma + u_F \sigma_F = 0,$$

$$u_S + u_F = 1$$

This system has the solution

$$\begin{aligned} u_S &= \frac{\sigma_F}{\sigma_F - \sigma}, \\ u_F &= \frac{-\sigma}{\sigma_F - \sigma}, \end{aligned}$$

and leaves us with the value dynamics

$$dV = V \cdot \{u_S(\alpha + \delta) + u_F\alpha_F\}\, dt.$$

Absence of arbitrage now implies that we must have the equation

$$u_S(\alpha + \delta) + u_F\alpha_F = r,$$

with probability 1, for all $t$, and, substituting the expressions for $u_F$, $u_S$, $\alpha_F$ and $\sigma_F$ into this equation, we get the equation

$$\frac{\partial F}{\partial t} + (r - \delta)S\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.$$

The boundary value is obvious, so we have the following result.

**Proposition 11.7** *The pricing function F(t,s) of the claim* $\Phi\ (S_T)$ *solves the boundary value problem*

$$\begin{cases} \dfrac{\partial F}{\partial t} + (r - \delta)s\dfrac{\partial F}{\partial s} + \dfrac{1}{2}\sigma^2 s^2 \dfrac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

(11.22)

Applying the Feynman–Kač representation theorem immediately gives us a risk neutral valuation formula.

**Proposition 11.8 (Pricing equation)** *The pricing function has the representation*

$$F(t,s) = e^{-r(T-t)}E^Q_{t,s}[\Phi(S_T)],$$

*where the Q-dynamics of S are given by*

(11.23)

$$dS_t = (r - \delta[S_t])S_t dt + \sigma(S_t)S_t dW_t.$$

(11.24)

In contrast with the case of discrete dividends we see that the appropriate martingale measure in the dividend case differs from that of the no dividend case. It is left as an exercise to prove the following result.

**Proposition 11.9 (Risk neutral valuation)** *Under the martingale measure Q, the* **normalized gain process**

$$G^Z(t) = e^{-rt}S(t) + \int_0^t e^{-r\tau} dD(\tau)$$

*is a Q-martingale.*

Note that this property is quite reasonable from an economic point of view: in a risk neutral world today's stock price should be the expected value of all future discounted earnings which arise from holding the stock. In other words, we expect that

$$S(0) = E^{Q}\left[\int_0^k e^{-r\tau} dD(\tau) + e^{-rt} S(t)\right], \tag{11.25}$$

and in the exercises the reader is invited to prove this "cost of carry" formula.

As in the discrete case it is natural to analyze the pricing formulas for the special case when we have the standard Black–Scholes dynamics

$$dS = \alpha S dt + \sigma S d\bar{W}, \tag{11.26}$$

where $\alpha$ and $\sigma$ are constants. We also assume that the dividend function $\delta$ is a deterministic constant. This implies that the martingale dynamics are given by

$$dS = (r - \delta) S dt + \sigma S dW, \tag{11.27}$$

i.e. $S$ is geometric Brownian motion also under the risk adjusted probabilities. Again we denote the pricing function by $F_\delta$ in order to highlight the dependence upon the parameter $\delta$. It is now easy to prove the following result, which shows how to price derivatives for a dividend paying stock in terms of pricing functions for a nondividend case.

**Proposition 11.10** *Assume that the functions $\sigma$ and $\delta$ are constant. Then, with notation as above, we have*

$$F_\delta(t,s) = F_0(t, se^{-\delta(T-t)}). \tag{11.28}$$

## 11.2.2 The General Case

We now consider a more general dividend structure, which we will need when dealing with futures contracts in Chapter 20 below.

**Assumption 11.2.2** *The price dynamics, under the objective probability measure, are given by*

$$dS_t = S_t \cdot \alpha(S_t) dt + S_t \cdot \sigma(S_t) d\bar{W}_t. \tag{11.29}$$

*The dividend structure is assumed to be of the form*

$$dD(t) = S_t \cdot \delta[S_t] dt + S_t \gamma[S_t] d\bar{W}_t, \tag{11.30}$$

*where $\delta$ and $\gamma$ are continuous deterministic functions.*

We again consider the pricing problem for a contingent $T$-claim of the form

$$\chi = \Phi(S_T),$$

and the only difference from the continuous yield case is that now we have to assume that the pricing function for the claim is a function of $D$ as well as $S$. We thus assume a claim price process of the form

$$\Pi(t, \chi) = F(t, S_t, D_t),$$

and then we carry out the standard program 1–7 of the previous section.

After a large number of simple, but messy and extremely boring, calculations which (needless to say) are left as an exercise, we end up with the following result.

**Proposition 11.11 (Pricing equation)** *The pricing function F(t,s,D) for the claim* $\chi = \Phi(S_T)$ *solves the boundary value problem*

$$\begin{cases} \frac{\partial F}{\partial t} + AF - rF &= 0, \\ F(T,s,D) &= \Phi(s), \end{cases} \tag{11.31}$$

*where*

$$\begin{aligned} AF &= \left(\frac{\alpha\gamma + \sigma r - \delta\sigma}{\sigma + \gamma}\right) s \frac{\partial F}{\partial s} + \left(\frac{\delta\sigma + \gamma r - \gamma\alpha}{\sigma + \gamma}\right) s \frac{\partial F}{\partial D} \\ &+ \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} + \frac{1}{2}\gamma^2 s^2 \frac{\partial^2 F}{\partial D^2} + \sigma\gamma s^2 \frac{\partial^2 F}{\partial s \partial D}. \end{aligned}$$

Using the Feynman–Kač technique we have the following risk neutral valuation result.

**Proposition 11.12 (Risk neutral valuation)** *The pricing function has the representation*

$$F(t,s,D) = e^{-r(T-t)} E^Q_{t,s,D}\left[\Phi(S_T)\right],$$

*where the Q-dynamics of S and D are given by*

$$\begin{aligned} dS_t &= S_t\left(\frac{\sigma\gamma + \sigma r - \delta\sigma}{\sigma + \gamma}\right)dt + S_t\sigma dW_t, \\ dD_t &= S_t\left(\frac{\delta\sigma + \gamma r - \gamma\alpha}{\sigma + \gamma}\right)dt + S_t\gamma dW_t. \end{aligned}$$

**Remark 11.2.1** In the expressions of the propositions above we have suppressed $S_t$ and $s$ in the functions α, σ, δ, γ.

The role of the martingale measure is the same as in the previous section.

**Proposition 11.13** *The martingale measure Q is characterized by the following facts*

- *There exists a market price of risk process λ such that the Q-dynamics are in the form*

$$\begin{aligned} dS &= S(\alpha - \lambda\sigma)dt + S\sigma dW, \\ dD &= S(\delta - \lambda\gamma)dt + S\gamma dW. \end{aligned}$$

- *The normalized gains process* $G_Z$, *defined by*

$$Z_t = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_r} dD(\tau),$$

*is a Q-martingale.*

This result has extensions to multidimensional factor models. We will not go into details, but are content with stating the main result.

**Proposition 11.14** *Consider a general factor model of the form in Sections10.3–10.4. If the market is free of arbitrage, then there will exist universal market price of risk processes* $\lambda = (\lambda_1, \ldots, \lambda_k)^*$ *such that*

- *For any T-claim* $\chi$ *the pricing function F has the representation*

$$F(t,x) = E^Q_{t,x}\left[e^{-\int_t^T r(X(u))du} \cdot \Phi(X(T))\right].$$

(11.32)

- *The Q-dynamics of the factor processes* $X^1, \ldots, X^k$ *are of the form*

$$dX^i = \{\mu_i - \delta_i \lambda\}\, dt + \delta_i dW,\ i = 1, \ldots, k.$$

- *For any price process S (underlying or derivative) with dividend process D, the normalized gains process*

$$Z_t = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_\tau} dD(\tau)$$

*is a Q-martingale.*

## 11.3 Exercises

**Exercise 11.1** Prove Proposition 11.1.

**Exercise 11.2** Prove the cost of carry formula (11.25).

**Exercise 11.3** Derive a cost-of-carry formula for the case of discrete dividends.

**Exercise 11.4** Prove Proposition 11.9.

**Exercise 11.5** Prove Proposition 11.10.

**Exercise 11.6** Consider the Black–Scholes model with a constant continuous dividend yield δ. Prove the following put–call parity relation, where $c_\delta$ ($p_\delta$) denotes the price of a European call (put).

$$p\delta = c\delta - se^{-\delta(T-t)} + Ke^{-r(T-t)}.$$

**Exercise 11.7** Consider the Black–Scholes model with a constant discrete dividend, as in eqns (11.17)–(11.18). Derive the relevant put–call parity for this case, given that there are $n$ remaining dividend points.

**Exercise 11.8** Consider the Black–Scholes model with a constant continuous dividend yield δ. The object of this exercise is to show that this model is complete. Take therefore as given a contingent claim $\chi = \Phi(S(T))$. Show that this claim can be replicated by a self-financing portfolio based on $B$ and $S$, and that the portfolios weights are given by

$$\begin{aligned} u^B(t,s) &= \frac{F(t,s) - sF_s(t,s)}{F(t,s)}, \\ u^S(t,s) &= \frac{sF_s(t,s)}{F(t,s)}, \end{aligned}$$

where $F$ is the solution of the pricing eqn (11.8).

**Hint:** Copy the reasoning from Chapter 7, while using the self-financing dynamics given in Section 5.3.

**Exercise 11.9** Consider the Black–Scholes model with a constant continuous dividend yield $\delta$. Use the result from the previous exercise in order to compute explicitly the replicating portfolio for the claim $\Phi\,(S(T)) = S(T)$.

**Exercise 11.10** Check that, when $\gamma = 0$ in Section 11.2.2, all results degenerate into those of Section 11.2.1.

**Exercise 11.11** Prove Propositions 11.11–11.13.

# 12 Currency Derivatives

In this chapter we will study a model which incorporates not only the usual domestic equity market, but also a market for the exchange rate between the domestic currency and a fixed foreign currency, as well as a foreign equity market. Financial derivatives defined in such situations are commonly known as **quanto products** . We will start by studying derivatives written directly on the exchange rate $X$, and then go on to study how to price (in the domestic currency) contracts written on foreign equity.

## 12.1 Pure Currency Contracts

Consider a situation where we have two currencies: the domestic currency (say pounds sterling), and the foreign currency (say US dollars). The spot exchange rate at time $t$ is denoted by $X(t)$, and by definition it is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}},$$

i.e. in our example it is quoted as pounds per dollar. We assume that the domestic short rate $r_d$, as well as the foreign short rate $r_f$, are deterministic constants, and we denote the corresponding riskless asset prices by $B_d$ and $B_f$ respectively. Furthermore we assume that the exchange rate is modelled by geometric Brownian motion. We can summarize this as follows. **Assumption 12.1.1** *We take as given the following dynamics (under the objective probability measure P)*

$$dX = X\alpha_X dt + X\sigma_X d\overline{W}, \tag{12.1}$$

$$dB_d = r_d B_d dt, \tag{12.2}$$

$$dB_f = r_f B_f dt, \tag{12.3}$$

*where $\alpha_X$, $\sigma_X$ are deterministic constants, and W is a scalar Wiener process.*

Our problem is that of pricing a currency derivative, i.e. a $T$-claim $Z$ of the form

$$Z = \Phi(X(T)),$$

where $\Phi$ is some given deterministic function. To take a concrete and important example, we can consider the case when $Z = \max[X(T) - K, 0]$ , i.e. we have a

European call which gives the owner the option to buy one unit of the foreign currency at the price $K$ (in the domestic currency).

At first glance it may perhaps seem that the problem of pricing the call option above is solved by use of the standard Black–Scholes formula, where we use domestic rate $r_d$ as the short rate of interest, and the stock price $S$ is replaced by the exchange rate $X$. It is, however, important to understand that this line of argument is incorrect, and the reason is as follows. When we buy a stock (without dividends), this means that we buy a piece of paper, which we keep until we sell it. When we buy a foreign currency (say US dollars) we will, on the contrary, not just keep the physical dollar bills until we sell them again. Instead we will typically put the dollars into an account where they will grow at a certain rate of interest. The obvious implication of this fact is that a foreign currency plays very much the same role as a **domestic stock with a continuous dividend** , and we will show below that this is indeed the case. First we formalize the institutional assumptions.

**Assumption 12.1.2** *All markets are frictionless and liquid. All holdings of the foreign currency are invested in the foreign riskless asset, i.e. they will evolve according to the dynamics*

$$dB_f = r_f B_f dt.$$

**Remark 12.1.1** Interpreted literally this means that, for example, US dollars are invested in a US bank. In reality this does not have to be the case—US dollars bought in Europe will typically be placed in a European so called **Eurodollar account** where they will command the Eurodollar rate of interest.

Applying the standard theory of derivatives to the present situation we have the usual risk neutral valuation formula

$$\Pi(t, Z) = e^{-r_d(T-t)} E_{t,x}^{Q} [\Phi(X(T))],$$

and our only problem is to figure out what the martingale measure $Q$ looks like. To do this we use the result from Proposition 9.3, that $Q$ is characterized by the property that every **domestic** asset has the short rate $r_d$ as its local rate of return under $Q$. In order to use this characterization we have to translate the possibility of investing in the foreign riskless asset into domestic terms. Since $B_f(t)$ units of the foreign currency are worth $B_f(t) \cdot X(t)$ in the domestic currency we immediately have the following result.

**Lemma 12.1** *The possibility of buying the foreign currency, and investing it at the foreign short rate of interest, is equivalent to the possibility of investing in a* **domestic** *asset with price process* $\tilde{B}_f$, *where*

$$\tilde{B}_f(t) = B_f(t) \cdot X(t).$$

*The dynamics of* $\tilde{B}_f$ *are given by*

$$d\tilde{B}_f = \tilde{B}_f(\alpha_X + r_f)dt + \tilde{B}_f \sigma_X d\bar{W}.$$

Summing up we see that our currency model is equivalent to a model of a domestic market consisting of the assets $B_d$ and $\tilde{B}_f$. It now follows directly from the general results that the martingale measure $Q$ has the property that the $Q$-dynamics of $\tilde{B}_f$ are given by

$$d\tilde{B}_f = r_d \tilde{B}_f dt + \tilde{B}_f \sigma_X dW, \tag{12.4}$$

where $W$ is a $Q$-Wiener process. Since by definition we have

$$X(t) = \frac{\tilde{B}_f}{B_f(t)}, \tag{12.5}$$

we can use Itô's formula, (12.2) and (12.4) to obtain the $Q$-dynamics of $X$ as

$$dX = X(r_d - r_f)dt + X\sigma_X dW. \tag{12.6}$$

The basic pricing result follows immediately.

**Proposition 12.2 (Pricing formulas)** *The arbitrage free price* $\Pi(t;\Phi)$ *for the T-claim* $Z = \Phi(X(T))$ *is given by* $\Pi(t;\Phi)=F(t, X(t))$, *where*

$$F(t,x) = e^{-r_d(T-t)} E^Q_{t,x}[\Phi(X(T))], \tag{12.7}$$

*and where the $Q$-dynamics of $X$ are given by*

$$dX = X(r_d - r_f)dt + X\sigma_X dW. \tag{12.8}$$

*Alternatively* $F(t, x)$ *can be obtained as the solution to the boundary value problem*

$$\begin{cases} \dfrac{\partial F}{\partial t} + x(r_d - r_f)\dfrac{\partial F}{\partial x} + \dfrac{1}{2}x^2\sigma_X^2\dfrac{\partial^2 F}{\partial x^2} - r_d F &= 0, \\ F(T,x) &= \Phi(x). \end{cases} \tag{12.9}$$

**Proof** The risk neutral valuation formula (12.7)–(12.8) follows from the standard risk neutral valuation formula and (12.6). The PDE result then follows via Feynman–Kač.

Comparing (12.8) to (11.27) we see that our original guess was correct: a foreign currency is to be treated exactly as a stock with a continuous dividend. We may thus draw upon the results from Section 11.2 (see Proposition 11.10), which allows us to use pricing formulas for stock prices (without dividends) to price currency derivatives.

**Proposition 12.3 (Option pricing formula)** *Let* $F_0(t, x)$ *be the pricing function for the claim* $Z = \Phi(X(T))$, *in a world where we interpret* $X$ *as the price of an ordinary stock without dividends. Let* $F(t, x)$ *be the pricing function of* *the same claim when X is interpreted as an exchange rate. Then the following relation holds*

$$F(t,x) = F_0(t, xe^{-r_f(T-t)}).$$

*In particular, the price of the European call,* $Z = \max[X(T) - K, 0]$ *, on the foreign currency, is given by the modified Black–Scholes formula*

$$F(t,x) = xe^{-r_f(T-t)}N[d_1] - e^{-r_d(T-t)}KN[d_2],$$

*where*

$$\begin{aligned} d_1(t,x) &= \frac{1}{\sigma_X\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma_X^2\right)(T-t)\right\}, \\ d_2(t,x) &= d_1(t,x) - \sigma_X\sqrt{T-t}. \end{aligned} \tag{12.10}$$

## 12.2 Domestic and Foreign Equity Markets

In this section we will model a market which, apart from the objects of the previous section, also includes a domestic equity with (domestic) price $S_d$, and a foreign equity with (foreign) price $S_f$. The restriction to a single domestic and foreign equity is made for notational convenience, and in most practical cases it is also sufficient.

We model the equity dynamics as geometric Brownian motion, and since we now have three risky assets we use a three-dimensional Wiener process in order to obtain a complete market.

**Assumption 12.2.1** *The dynamic model of the entire economy, under the objective measure P, is as follows.*

$$dX = X\alpha_X dt + X\sigma_X d\bar{W},$$

$$dS_d = S_d\alpha_d dt + S_d\sigma_d d\bar{W}, \tag{12.11}$$

$$dS_f = S_f\alpha_f dt + S_f\sigma_f d\bar{W}, \tag{12.12}$$

$$dB_d = r_d B_d dt, \tag{12.13}$$

$$dB_f = r_f B_f dt, \tag{12.14}$$

*where*

$$\bar{W} = \begin{bmatrix} \bar{W}_1 \\ \bar{W}_2 \\ \bar{W}_3 \end{bmatrix} \tag{12.15}$$

*is a three-dimensional Wiener process (as usual with independent components). Furthermore, the* $(3\times 3)$*-dimensional matrix* $\sigma$*, given by*

$$\sigma = \begin{bmatrix} \sigma_X \\ \sigma_d \\ \sigma_f \end{bmatrix} = \begin{bmatrix} \sigma_{X1} & \sigma_{X2} & \sigma_{X3} \\ \sigma_{d1} & \sigma_{d2} & \sigma_{d3} \\ \sigma_{f1} & \sigma_{f2} & \sigma_{f3} \end{bmatrix},$$

*is assumed to be invertible.*

**Remark 12.2.1** The reason for the assumption about $\sigma$ is that this is the necessary and sufficient condition for completeness. See Proposition 12.4 below.

**Remark 12.2.2** It is also possible, and in many situations convenient, to model the market using three scalar correlated Wiener processes (one for each asset). See Remark 12.2.4 below.

Typical $T$-contracts which we may wish to price (in terms of the domestic currency) are given by the following list.

- A **foreign equity call, struck in foreign currency** , i.e. an option to buy one unit of the foreign equity at the strike price of $K$ units of the foreign currency. The value of this claim at the date of expiration is, expressed in the foreign currency, given by

$$Z^f = \max\left[S_f(T) - K, 0\right]. \tag{12.16}$$

Expressed in terms of the domestic currency the value of the claim at $T$ is

$$Z^d = X(T) \cdot \max\left[S_f(T) - K, 0\right]. \tag{12.17}$$

- A **foreign equity call, struck in domestic currency** , i.e. a European option to buy one unit of the foreign equity at time $T$, by paying $K$ units of the domestic currency. Expressed in domestic terms this claim is given by

$$Z^d = \max\left[X(T) \cdot S_f(T) - K, 0\right]. \tag{12.18}$$

- An **exchange option** which gives us the right to exchange one unit of the domestic equity for one unit of the foreign equity. The corresponding claim, expressed in terms of the domestic currency, is

$$Z^d = \max\left[X(T) \cdot S_f(T) - S_d(T), 0\right]. \tag{12.19}$$

More generally we will study pricing problems for $T$-claims of the form

$$Z = \Phi(X(T), S_d(T), S_f(T)), \tag{12.20}$$

where $Z$ is measured in the domestic currency. From the general theory of Chapter 9 we know that the pricing function $F(t, x, s_d, s_f)$ is given by the risk neutral valuation formula

$$F(t, x, s_d, s_f) = e^{-r_d(T-t)} E^Q_{t,x,s_d,s_f}\left[\Phi(X(T))\right],$$

so we only have to find the correct risk adjusted measure $Q$. We follow the technique of the previous section and transform all foreign traded assets into domestic terms. The foreign bank account has already been taken care of, and it is obvious that one unit of the foreign stock, worth $S_f(t)$ in the foreign currency, is worth $X(t) \cdot S_f(t)$ in domestic terms. We thus have the following equivalent domestic model, where the asset dynamics follow from the Itô formula.

**Proposition 12.4** *The original market* (12.11)–(12.15) *is equivalent to a market consisting of the price processes* $S_d$, $\tilde{S}_f$, $\tilde{B}_f$, $B_d$, *where*

$$\tilde{B}_f(t) = X(t)B_f(t),$$

$$\tilde{S}_f(t) = X(t)S_f(t).$$

*The P-dynamics of this equivalent model are given by*

$$dS_d = S_d\alpha_d dt + S_d\sigma_d d\bar{W},$$

$$d\tilde{S}_f = \tilde{S}_f(\alpha_f + \alpha_X + \sigma_f\sigma_X^*)dt + \tilde{S}_f(\sigma_f + \sigma_X)d\bar{W}, \tag{12.21}$$

$$d\tilde{B}_f = \tilde{B}_f(\alpha_X + r_f)dt + \tilde{B}_f\sigma_X d\bar{W}, \tag{12.22}$$

$$dB_d = r_d B_d dt. \tag{12.23}$$

(12.24)

Here we have used $^*$ to denote transpose, so

$$\sigma_f\sigma_X^* = \sum_{i=1}^{3}\sigma_{fi}\sigma_{Xi}.$$

Note that, because of Assumption 12.2.1, the volatility matrix above is invertible, so the market is complete.

Since $S_d$, $\tilde{S}_f$, $\tilde{B}_f$ can be interpreted as prices of domestically traded assets, we can easily obtain the relevant $Q$-dynamics.

**Proposition 12.5** *The Q-dynamics are as follows.*

$$dS_d = S_d r_d dt + S_d\sigma_d dW,$$

$$d\tilde{S}_f = \tilde{S}_f r_d dt + \tilde{S}_f(\sigma_f + \sigma_X)dW, \tag{12.25}$$

$$d\tilde{B}_f = \tilde{B}_f r_d dt + \tilde{B}_f\sigma_X dW, \tag{12.26}$$

$$dX = X(r_d - r_f)dt + X\sigma_X dW, \tag{12.27}$$

$$dS_f = S_f(r_f - \sigma_f\sigma_X^*)dt + S_f\sigma_f dW. \tag{12.28}$$

(12.29)

**Proof** Equations (12.25)–(12.27) follow from Proposition 9.3. The equation for $X$ follows from (12.27), the relation $X=B^f/B_f$ and Itô's formula. A similar computation applied to the relation $S_f=\tilde{S}_f/X$ gives us (12.29).

We can now immediately obtain the risk neutral valuation formula, and this can in fact be done in two ways. We can either use $(X, S_d, S_f)$ as state variables, or use the equivalent (there is a one-to-one mapping) set $(X, S_d, \tilde{S}_f)$. Which set to use is a matter of convenience, depending on the particular claim under study, but in both cases the arbitrage free price is given by the discounted expected value of the claim under the $Q$-dynamics. (It is of course quite possible to use the set $(\tilde{B}_f, S_d, \tilde{S}_f)$ also, but there seems to be no point in doing so.)

**Proposition 12.6 (Pricing formulas)** *For a claim of the form*

$$Z = \Phi(X(T), S_d(T), \tilde{S}_f(T))$$

*the corresponding pricing function* $F(t, x, s_d, \tilde{s}_f)$ *is given by*

$$F(t, x, s_d, \tilde{s}_f) = e^{-r_d(T-t)} E^Q_{t,x,s_d,\tilde{s}_f}\left[\Phi(X(T), S_d(T), \tilde{S}_f(T))\right],$$

*where the Q-dynamics are given by Proposition 12.5.* (12.30)

*The pricing PDE is*

$$\begin{aligned}
&\frac{\partial F}{\partial t} + x(r_d - r_f)\frac{\partial F}{\partial x} + s_d r_d \frac{\partial F}{\partial s_d} + \tilde{s}_f r_d \frac{\partial F}{\partial \tilde{s}_f} \\
&+\frac{1}{2}\left\{x^2 \left\|\sigma_X\right\|^2 \frac{\partial^2 F}{\partial x^2} + s_d^2 \left\|\sigma_d\right\|^2 \frac{\partial^2 F}{\partial s_d^2} + \tilde{s}_f^2 \left(\left\|\sigma_f\right\|^2 + \left\|\sigma_X\right\|^2 + 2\sigma_f \sigma_X^*\right) \frac{\partial^2 F}{\partial \tilde{s}_f^2}\right\} \\
&+s_d x \sigma_d \sigma_X^* \frac{\partial^2 F}{\partial s_d \partial x} + \tilde{s}_f x \left(\sigma_f \sigma_X^* + \left\|\sigma_X\right\|^2\right) \frac{\partial^2 F}{\partial \tilde{s}_f \partial x} \\
&+s_d \tilde{s}_f (\sigma_d \sigma_f^* + \sigma_d \sigma_X^*) \frac{\partial^2 F}{\partial s_d \partial \tilde{s}_f} \\
&-r_d F = 0, \\
&F(T, x, s_d, \tilde{s}_f) = \Phi(x, s_d, \tilde{s}_f).
\end{aligned}$$

**Proposition 12.7** *For a claim of the form*

$$Z = \Phi(X(T), S_d(T), S_f(T))$$

*the corresponding pricing function* $F(t, x, s_d, s_f)$ *is given by*

$$F(t, x, s_d, s_f) = e^{-r_d(T-t)} E^Q_{t,x,s_d,s_f}\left[\Phi(X(T), S_d(T), S_f(T))\right],$$

*where the Q-dynamics are given by Proposition 12.5.* (12.31)

*The pricing PDE is*

$$\begin{aligned}
&\frac{\partial F}{\partial t} + x(r_d - r_f)\frac{\partial F}{\partial x} + s_d r_d \frac{\partial F}{\partial s_d} + s_f(r_f - \sigma_f \sigma_X^*)\frac{\partial F}{\partial s_f} \\
&+\frac{1}{2}\left(x^2 \left\|\sigma_X\right\|^2 \frac{\partial^2 F}{\partial x^2} + s_d^2 \left\|\sigma_d\right\|^2 \frac{\partial^2 F}{\partial s_d^2} + s_f^2 \left\|\sigma_f\right\|^2 \frac{\partial^2 F}{\partial s_f^2}\right) \\
&+s_d x \sigma_d \sigma_X^* \frac{\partial^2 F}{\partial s_d \partial x} + s_f x \sigma_f \sigma_X^* \frac{\partial^2 F}{\partial s_f \partial x} + s_d s_f \sigma_d \sigma_f^* \frac{\partial^2 F}{\partial s_d \partial s_f} \\
&-r_d F = 0, \\
&F(T, x, s_d, s_f) = \Phi(x, s_d, s_f).
\end{aligned}$$

**Remark 12.2.3** In many applications the claim under study is of the restricted form

$$Z = \Phi(X(T), S_f(T)).$$

In this case all partial derivatives w.r.t. $s_d$ vanish from the PDEs above. A similar reduction will of course also take place for a claim of the form

$$Z = \Phi(X(T), S_d(T)).$$

We end this section by pricing the contracts (12.16) and (12.18) above.

**Example 12.8 (Foreign call, struck in foreign currency)** We recall that the value of the claim, at time $T$, expressed in the foreign currency, is

$$Z^f = \max\left[S_f(T) - K, 0\right].$$

Let us denote the pricing function, again in the foreign currency, for this claim at time $t$, by $F^f(t, s_f)$. (It will obviously not involve $x$.) Furthermore we denote value of the claim at time $t$, expressed in the domestic currency, by $F^d(t, x, s_f)$. Now an elementary arbitrage argument (which?) immediately gives us the relation

$$F^d(t, x, s_f) = x \cdot F^f(t, s_f),$$

so it only remains to compute $F^f(t, s_f)$. This, however, is just the value of a European call on a stock with volatility $\| \sigma_f \|$, in an economy with a short rate equal to $r_f$. Thus the value is given by the Black–Scholes formula, and the pricing formula in domestic terms is as follows.

$$F^d(t, x, s_f) = x s_f N[d_1] - x e^{-r_f(T-t)} K N[d_2],$$

where

$$\begin{aligned} d_1(t, s_f) &= \frac{1}{\|\sigma_f\|\sqrt{T-t}}\left\{\ln\left(\frac{s_f}{K}\right) + \left(r_f + \frac{1}{2}\|\sigma_f\|^2\right)(T-t)\right\}, \\ d_2(t, s_f) &= d_1(t, s_f) - \|\sigma_f\|\sqrt{T-t}. \end{aligned}$$

See also Remark 12.2.4 for another formalism.

**Example 12.9 (Foreign call, struck in domestic currency)** The claim, expressed in domestic terms, is given by

$$Z^d = \max\left[X(T) \cdot S_f(T) - K, 0\right]$$

which we write as

$$Z^d = \max\left[\tilde{S}_f(T) - K, 0\right].$$

As we have seen above, the process $\tilde{S}_f$ can be interpreted of the price process of a domestically traded asset, and from Proposition 12.5 we see that its volatility is given by $\| \sigma_f + \sigma_X \|$. Thus we may again use the Black–Scholes formula to obtain the pricing function $F(t, \tilde{s}_f)$ as

$$F(t,\tilde{s}_f) = \tilde{s}_f N[d_1] - e^{-r_d(T-t)} K N[d_2],$$

where

$$\begin{aligned} d_1(t,\tilde{s}_f) &= \frac{1}{\left\|\sigma_f + \sigma_X\right\|\sqrt{T-t}} \left\{ \ln\left(\frac{\tilde{s}_f}{K}\right) + \left(r_d + \frac{1}{2}\left\|\sigma_f + \sigma_X\right\|^2\right)(T-t) \right\}, \\ d_2(t,\tilde{s}_f) &= d_1(t,\tilde{s}_f) - \left\|\sigma_f + \sigma_X\right\|\sqrt{T-t}. \end{aligned}$$

See also Remark 12.2.4 for another formalism.

**Remark 12.2.4** In practical applications it may be more convenient to model the market, and easier to read the formulas above, if we model the market using correlated Wiener processes (see Section 3.8). We can formulate our basic model (under $Q$) as

$$\begin{aligned} dX &= X\alpha_X dt + X\delta_X d\bar{V}_X, \\ dS_d &= S_d\alpha_d dt + S_d\delta_d d\bar{V}_d, \\ dS_f &= S_f\alpha_f dt + S_f\delta_f d\bar{V}_f, \\ dB_d &= r_d B_d dt, \\ dB_f &= r_f B_f dt, \end{aligned}$$

where the three processes $\bar{V}_X$, $\bar{V}_d$, $\bar{V}_f$ are one-dimensional correlated Wiener processes. We assume that $\delta_X,\delta_d,\delta_f$ are positive. The instantaneuos correlation between $\bar{V}_X$ and $\bar{V}_f$ is denoted by $\varrho_{Xf}$ and correspondingly for the other pairs. We then have the following set of translation rules between the two formalisms

$$\begin{aligned} \|\sigma_i\| &= \delta_i, & i &= X,d,f, \\ \sigma_i\sigma_j^* &= \delta_i\delta_j\rho_{ij}, & i,j &= X,d,f, \\ \|\sigma_i + \sigma_j\| &= \sqrt{\delta_i^2 + \delta_j^2 + 2\delta_i\delta_j\rho_{ij}}, & i,j &= X,d,f. \end{aligned}$$

## 12.3 Domestic and Foreign Market Prices of Risk

This section constitutes a small digression in the sense that we will not derive any new pricing formulas. Instead we will take a closer look at the various market prices of risk. As will be shown below, we have to distinguish between the domestic and the foreign market price of risk, and we will clarify the connection between these two objects. As a by-product we will obtain a somewhat deeper understanding of the concept of risk neutrality.

Let us therefore again consider the international model $X$, $S_d$, $S_f$, $B_d$, $B_f$, with dynamics under the objective measure $P$ given by (12.11)–(12.15). As before we transform the international model into the domestically traded assets $S_d$, $\tilde{S}_f$, $B_d$, $\tilde{B}_f$ with $P$-dynamics given by (12.21)–(12.24).

In the previous section we used the general results from Chapter 9 to infer the existence of a martingale measure $Q$, under which all **domestically** traded assets command the domestic short rate $r_d$ as the local rate of return. Our first observation is that, from a logical point of view, we could just as well have chosen to transform (12.11)–(12.15) into equivalent assets traded on the **foreign** market. Thus we should really denote our "old" martingale measure $Q$ by $Q_d$ in order to emphasize its dependence on the domestic point of view. If we instead take a foreign investor's point of view we will end up with a "foreign martingale measure", which we will denote by $Q_f$, and an obvious project is to investigate the relationship between these martingale measures. A natural guess is perhaps that $Q_d = Q_f$, but as we shall see this is generically not the case. Since there is a one-to-one correspondence between martingale measures and market prices of risk, we will carry out the project above in terms of market prices of risk.

We start by taking the domestic point of view, and applying Result 10.5.1 to the domestic price processes (12.21)–(12.24), we infer the existence of the **domestic market price of risk** process

$$\lambda_d(t) = \begin{bmatrix} \lambda_{d1}(t) \\ \lambda_{d2}(t) \\ \lambda_{d3}(t) \end{bmatrix}$$

with the property that if $\Pi$ is the price process of any domestically traded asset in the model, with price dynamics under $P$ of the form

$$d\Pi(t) = \Pi(t)\alpha_\Pi(t)dt + \Pi(t)\sigma_\Pi(t)d\bar{W}(t),$$

then, for all $t$ and $P$-a.s., we have

$$\alpha_\Pi(t) - r_d = \sigma_\Pi(t)\lambda_d(t).$$

Applying this to (12.21)–(12.23) we get the following set of equations

$$\alpha_d - r_d = \sigma_d \cdot \lambda_d, \tag{12.32}$$

$$\alpha_f + \alpha_X + \sigma_f \sigma_X^\star - r_d = (\sigma_f + \sigma_X)\lambda_d, \tag{12.33}$$

$$\alpha_X + r_f - r_d = \sigma_X \cdot \lambda_d. \tag{12.34}$$

In passing we note that, since the coefficient matrix

$$\sigma = \begin{bmatrix} \sigma_d \\ \sigma_f + \sigma_X \\ \sigma_X \end{bmatrix}$$

is invertible by Assumption 12.2.1, $\lambda_d$ is uniquely determined (and in fact constant). This uniqueness is of course equivalent to the completeness of the model.

We now go on to take the perspective of a foreign investor, and the first thing to notice is that the model (12.11)–(12.15) of the international market does not treat the foreign and the domestic points of view symmetrically. This is due to the fact that the exchange rate $X$ by definition is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}}.$$

From the foreign point of view the exchange rate $X$ should thus be replaced by the exchange rate

$$Y(t) = \frac{1}{X(t)}$$

which is then quoted as

$$\frac{\text{unit of the foreign currency}}{\text{units of the domestic currency}},$$

and the dynamics for $X$, $S_d$, $S_f$, $B_d$, $B_f$ should be replaced by the dynamics for $Y$, $S_d$, $S_f$, $B_d$, $B_f$. In order to do this we only have to compute the dynamics of $Y$, given those of $X$, and an easy application of Itô's formula gives us

$$dY = Y\alpha_Y dt + Y\sigma_Y d\bar{W},$$

where

$$\alpha_Y = -\alpha_X + \|\sigma_X\|^2, \tag{12.35}$$

$$\sigma_Y = -\sigma_X. \tag{12.36}$$

(12.37)

Following the arguments from the domestic analysis, we now transform the processes $Y$, $S_d$, $S_f$, $B_d$, $B_f$ into a set of asset prices on the foreign market, namely $S_f$, $\tilde{S}_d$, $\tilde{B}_d$, where

$$\begin{aligned} \tilde{S}_d &= Y \cdot S_d, \\ \tilde{B}_d &= Y \cdot B_d. \end{aligned}$$

If we want to obtain the $P$-dynamics of $S_f$, $\tilde{S}_d$, $\tilde{B}_d$ we now only have to use (12.21)–(12.24), substituting $Y$ for $X$ and $_d$ for $_f$. Since we are not interested in these dynamics *per se*, we will, however, not carry out these computations. The object that we are primarily looking for is the foreign market price of risk $\lambda_f$, and we can easily obtain that by writing down the foreign version of the system and substituting $_d$ for $_f$ and $Y$ for $X$ directly in (12.32)–(12.34). We get

$$\begin{aligned} \alpha_f - r_f &= \sigma_f \cdot \lambda_f, \\ \alpha_d + \alpha_Y + \sigma_d \sigma_Y^{*} - r_f &= (\sigma_d + \sigma_Y)\lambda_f, \\ \alpha_Y + r_d - r_f &= \sigma_Y \cdot \lambda_f, \end{aligned}$$

and, inserting (12.36)–(12.37), we finally obtain

$$\alpha_f - r_f = \sigma_f \cdot \lambda_f, \tag{12.38}$$

$$\alpha_d - \alpha_X + \|\sigma_X\|^2 - \sigma_d\sigma_X^* - r_f = (\sigma_d - \sigma_X)\lambda_f, \tag{12.39}$$

$$-\alpha_X + \|\sigma_X\|^2 + r_d - r_f = -\sigma_X \cdot \lambda_f. \tag{12.40}$$

After some simple algebraic manipulations, the two systems (12.32)–(12.34) and (12.38)–(12.40) can be written as

$$\begin{aligned} \alpha_X + r_f - r_d &= \sigma_X \cdot \lambda_d, \\ \alpha_d - r_d &= \sigma_d \cdot \lambda_d, \\ \alpha_f + \sigma_f\sigma_X^* - r_f &= \sigma_f \cdot \lambda_d, \end{aligned}$$

$$\begin{aligned} \alpha_X - \left\|\sigma_X\right\|^2 + r_f - r_d &= \sigma_X \cdot \lambda_f, \\ \alpha_d - \sigma_d\sigma_X^* - r_d &= \sigma_d \cdot \lambda_f, \\ \alpha_f - r_f &= \sigma_f \cdot \lambda_f. \end{aligned}$$

These equations can be written as

$$\begin{aligned} \delta &= \sigma\lambda_d, \\ \varphi &= \sigma\lambda_f, \end{aligned}$$

where

$$\delta = \begin{bmatrix} \alpha_X + r_f - r_d \\ \alpha_d - r_d \\ \alpha_f + \sigma_f\sigma_X^* - r_f \end{bmatrix}, \varphi = \begin{bmatrix} \alpha_X - \left\|\sigma_X\right\|^2 + r_f - r_d \\ \alpha_d - \sigma_d\sigma_X^* - r_d \\ \alpha_f - r_f \end{bmatrix}$$

so, since $\sigma$ is invertible,

$$\begin{aligned} \lambda_d &= \sigma^{-1}\delta, \\ \lambda_f &= \sigma^{-1}\varphi. \end{aligned}$$

Thus we have

$$\lambda_d - \lambda_f = \sigma^{-1}(\delta - \varphi),$$

and since

$$\delta - \varphi = \begin{bmatrix} \sigma_X\sigma_X^* \\ \sigma_d\sigma_X^* \\ \sigma_f\sigma_X^* \end{bmatrix} = \sigma\sigma_X^*,$$

we obtain

$$\lambda_d - \lambda_f = \sigma^{-1}(\delta - \varphi) = \sigma^{-1}\sigma\sigma_X^* = \sigma_X^*.$$

We have thus proved the following central result.

**Proposition 12.10** *The foreign market price of risk is uniquely determined by the domestic market price of risk, and by the exchange rate volatility vector* $\sigma_X$, *through the formula*

$$\lambda_f = \lambda_d - \sigma_X^{\star}.$$

(12.41)

**Remark 12.3.1** For the benefit of the probabilist we note that this result implies that the transition from $Q_d$ to $Q_f$ is effected via a Girsanov transformation, for which the likelihood process $L$ has the dynamics

$$\begin{aligned} dL &= L\sigma_X dW, \\ L(0) &= 1. \end{aligned}$$

Proposition 12.10 has immediate consequences for the existence of risk neutral markets. If we focus on the domestic market can we say that the market is (on the aggregate) risk neutral if the following valuation formula holds, where $\Pi_d$ is the price process for any domestically traded asset.

$$\Pi_d(t) = e^{-r_d(T-t)} E^P [\Pi_d(T) | \mathcal{F}_t].$$

(12.42)

In other words, the domestic market is risk neutral if and only if $P=Q_d$. In many scientific papers an assumption is made that the domestic market is in fact risk neutral, and this is of course a behavioral assumption, typically made in order to facilitate computations. In an international setting it then seems natural to assume that both the domestic market **and** the foreign market are risk neutral, i.e. that, in addition to (12.42), the following formula also holds, where $\Pi_f$ is the foreign price of any asset traded on the foreign market

$$\Pi_f(t) = e^{-r_f(T-t)} E^P [\Pi_f(T) | \mathcal{F}_t].$$

(12.43)

This seems innocent enough, but taken together these assumptions imply that

$$P = Q_d = Q_f.$$

(12.44)

Proposition 12.10 now tells us that (12.44) **can never hold, unless** $\boldsymbol{\sigma_X = 0}$ , i.e. if and only if the exchange rate is deterministic.

At first glance this seems highly counter-intuitive, since the assumption about risk neutrality often is interpreted as an assumption about the (aggregate) attitude towards risk **as such** . However, from (12.42), which is an equation for objects measured in the domestic currency, it should be clear that risk neutrality is a property which holds only **relative to a specified numeraire** . To put it as a slogan, you may very well be risk neutral w.r.t. pounds sterling, and still be risk averse w.r.t. US dollars.

There is nothing very deep going on here: it is basically just the Jensen inequality. To see this more clearly let us consider the following simplified situation. We assume that $r_d=r_f=0$, and we assume that the domestic market is risk neutral. This means in particular that the exchange rate itself has the following risk neutral valuation formula

$$X(0) = E[X(T)].$$

(12.45)

Looking at the exchange rate from the foreign perspective we see that if the foreign market also is risk neutral, then it must hold that

$$Y(0) = E[Y(T)], \tag{12.46}$$

with $Y=1/X$. The Jensen inequality together with (12.45) gives us, however,

$$Y(0) = \frac{1}{X(0)} = \frac{1}{E[X(T)]} \leq E\left[\frac{1}{X(T)}\right] = E[Y(T)]. \tag{12.47}$$

Thus (12.45) and (12.46) can never hold simultaneously with a stochastic exchange rate.

## 12.4 Exercises

**Exercise 12.1** Consider the European call on the exchange rate described at the end of Section 12.1. Denote the price of the call by $c(t, x)$, and denote the price of the corresponding put option (with the same exercise price $K$ and exercise date $T$) by $p(t, x)$. Show that the put–call parity relation between $p$ and $c$ is given by

$$p = c - xe^{-r_f(T-t)} + Ke^{-r_d(T-t)}$$

**Exercise 12.2** Compute the pricing function (in the domestic currency) for a **binary option** on the exchange rate. This option is a $T$-claim, $Z$, of the form

$$Z = 1_{[a,b]}(X(T)),$$

i.e. if $a \leq X(T) \leq b$ then you will obtain one unit of the domestic currency, otherwise you get nothing.

**Exercise 12.3** Derive the dynamics of the domestic stock price $S_d$ under the foreign martingale measure $Q_f$.

**Exercise 12.4** Compute a pricing formula for the exchange option in (12.19). Use the ideas from Section 9.4 in order to reduce the complexity of the formula. For simplicity you may assume that the processes $S_d$, $S_f$ and $X$ are uncorrelated.

**Exercise 12.5** Consider a model with the domestic short rate $r_d$ and two foreign currencies, the exchange rates of which (from the domestic perspective) are denoted by $X_1$ and $X_2$ respectively. The foreign short rates are denoted by $r_1$ and $r_2$ respectively. We assume that the exchange rates have $P$-dynamics given by

$$dX_i = X_i \alpha_i dt + X_i \sigma_i d\bar{W}_i, \quad i = 1,2,$$

where $\bar{W}_1$, $\bar{W}_2$ are $P$-Wiener processes with correlation $\varrho$.

(a) Derive the pricing PDE for contracts, quoted in the domestic currency, of the form $Z = \Phi(X_1(T), X_2(T))$.
(b) Derive the corresponding risk neutral valuation formula, and the $Q_d$-dynamics of $X_1$ and $X_2$.

(c) Compute the price, in domestic terms, of the "binary quanto contract" $\mathcal{Z}$, which gives, at time $T$, $K$ units of foreign currency No. 1, if $a \leq X_2(T) \leq b$, (where $a$ and $b$ are given numbers), and zero otherwise. If you want to facilitate computations you may assume that $\varrho = 0$.

**Exercise 12.6** Consider the model of the previous exercise. Compute the price, in domestic terms, of a quanto exchange option, which at time $T$ gives you the option, but not the obligation, to exchange $K$ units of currency No. 1 for 1 unit of currency No. 2.

**Hint:** It is possible to reduce the state space as in Section 9.4.

## 12.5 Notes

The classic in this field is Garman and Kohlhagen (1983). See also Reiner (1992). A more technical treatment is given in Amin and Jarrow (1991).

# 13 Barrier Options

The purpose of this chapter is to give a reasonably systematic overview of the pricing theory for those financial derivatives which are, in some sense, connected to the extremal values of the underlying price process. We focus on barrier options, ladders and lookbacks, and we confine ourselves to the case of one underlying asset.

## 13.1 Mathematical Background

In this chapter we will give some probability distributions connected with barrier problems. All the results are standard, see e.g. Borodin–Salminen (1997).

To start with some notational conventions, let $\{X(t); \quad 0 \leq t < \infty\}$ be any process with continuous trajectories taking values on the real line.

**Definition 13.1** *For any* $y \in R$, *the* **hitting time** *of* $y$, $\tau(X, y)$, *sometimes denoted by* $\tau(y)$ *or* $\tau_y$, *is defined by*

$$\tau(y) = \inf \quad \{t \geq 0 \quad | X(t) = y\}.$$

*The X-process* **absorbed at** $y$ *is defined by*

$$X_y(t) = X(t \wedge \tau)$$

*where we have used the notation* $\alpha \wedge \beta = \min[\alpha, \beta]$.

*The* **running maximum** *and* **minimum** *processes,* $M_X(t)$ *and* $m_X(t)$, *are defined by*

$$\begin{aligned} M_X(t) &= \sup_{0 \leq s \leq t} X(s), \\ m_X(t) &= \inf_{0 \leq s \leq t} X(s), \end{aligned}$$

*where we sometimes suppress the subscript X.*

We will be mainly concerned with barrier problems for Wiener processes, so naturally the normal distribution will play a prominent role.

**Definition 13.2** *Let* $\varphi(x; \mu, \sigma)$ *denote the density of a normal distribution with mean* $\mu$ *and variance* $\sigma^2$, *i.e.*

$$\varphi(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}.$$

*The standardized density* $\varphi(x;0, 1)$ *is denoted by* $\varphi(x)$, *and the cumulative distribution function of* $\varphi(x)$ *is as usual denoted by* $N(x)$, *i.e.*

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz.$$

Let us now consider a Wiener process with (constant) drift $\mu$ and (constant) diffusion $\sigma$, starting at a point $\alpha$, i.e.

$$dX(t) = \mu dt + \sigma dW_t, \tag{13.1}$$

$$X(0) = \alpha. \tag{13.2}$$

We are primarily interested in the one-dimensional marginal distribution for $X_\beta(t)$, i.e. the distribution at time $t$ of the $X$-process, absorbed at the point $\beta$. The distribution of $X_\beta(t)$ is of course a mixed distribution in the sense that it has a point mass at $x = \beta$ (the probability that the process is absorbed prior to time $t$) and a density. This density has its support on the interval $(\beta, \infty)$ if $\alpha > \beta$, whereas the support is the interval $(-\infty, \beta)$ if $\alpha < \beta$. We now cite our main result concerning absorption densities.

**Proposition 13.3** *The density* $f_\beta(x;t, \alpha)$ *of the absorbed process* $X_\beta(t)$, *where* $X$ *is defined by* (13.1)–(13.2), *is given by*

$$f_\beta(x; t,\alpha) = \varphi(x; \mu t + \alpha, \sigma\sqrt{t}) - \exp\left\{-\frac{2\mu(\alpha-\beta)}{\sigma^2}\right\}\varphi(x; \mu t - \alpha + 2\beta, \sigma\sqrt{t}).$$

*The support of this density is the interval* $(\beta, \infty)$ *if* $\alpha > \beta$, *and the interval* $(-\infty, \beta)$ *if* $\alpha < \beta$.

We end this section by giving the distribution for the running maximum (minimum) processes.

**Proposition 13.4** *Consider the process* $X$ *defined by (13.1)–(13.2), and let* $M$ ($m$) *denote the running maximum (minimum) processes as in Definition 13.1. Then the distribution functions for* $M(t)$ ($m(t)$) *are given by the following expressions, which hold for* $x \geq \alpha$ *and* $x \leq \alpha$ *respectively.*

$$F_{M(t)}(x) = N\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) - \exp\left\{2\cdot\frac{\mu(x-\alpha)}{\sigma^2}\right\}N\left(-\frac{x-\alpha+\mu t}{\sigma\sqrt{t}}\right),$$

$$F_{m(t)}(x) = N\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) + \exp\left\{2\cdot\frac{\mu(x-\alpha)}{\sigma^2}\right\}N\left(\frac{x-\alpha+\mu t}{\sigma\sqrt{t}}\right)$$

# 13.2 Out Contracts

In this section we will undertake a systematic study of the relations between a "standard" contingent claim and its different "barrier" versions. This will provide us with some basic insights and will also give us a number of easy formulas to use when pricing various barrier contracts. As usual we consider the standard Black–Scholes model

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= rB dt, \end{aligned}$$

with fixed parameters α, σ and *r*.

We fix an exercise time *T* and we consider as usual a contingent claim $Z$ of the form

$$Z = \Phi(S(T)). \tag{13.3}$$

We denote the pricing function of $Z$ by $F(t,s;T,\Phi)$, often suppressing the parameter *T*. For mnemo-technical purposes we will also sometimes use the notation $\mathbf{\Phi}(t, s)$, i.e. the pricing function (as opposed to the function defining the claim) is given in bold.

## 13.2.1 Down-And-Out Contracts

Fix a real number $L < S(0)$, which will act as the barrier, and consider the following contract, which we denote by $Z_{LO}$:

- If the stock price stays above the barrier *L* during the entire contract period, then the amount $Z$ is paid to the holder of the contract.
- If the stock price, at some time before the delivery time *T*, hits the barrier *L*, then the contract ceases to exist, and nothing is paid to the holder of the contract.

The contract $Z_{LO}$ is called the "down-and-out" version of the contract $Z$ above, and our main problem is to price $Z_{LO}$. More formally we can describe $Z_{LO}$ as follows.

**Definition 13.5** *Take as given a T-contract* $Z = \Phi(S(T))$. *Then the T-contract* $Z_{LO}$ *is defined by*

$$Z_{LO} = \begin{cases} \Phi(S(T)), & \text{if} \quad S(t) > L \quad \text{for all} \quad t \in [0,T] \\ 0, & \text{if} \quad S(t) \le L \quad \text{for some} \quad t \in [0,T]. \end{cases} \tag{13.4}$$

Concerning the notation, *L* as a **sub** script indicates a "down"-type contract, whereas the letter *O* indicates that we are considering an "out" claim. You may also consider other types of barrier specifications and thus construct a "down-and-in" version of the basic contract $Z$. A "down-and-in" contract **starts** to exist the first time the stock price hits a lower barrier. Going on we may then consider up-and-out as well as up-and-in contracts. All these types will be given precise definitions and studied in the following sections.

In order to price $Z_{LO}$ we will have use for the function $\Phi_L$, which is the original contract function Φ in (13.3) "chopped off" below *L*.

**Definition 13.6** *For a fixed function* Φ *the function* $\Phi_L$ *is defined by*

$$\Phi_L(x) = \begin{cases} \Phi(x), & \text{for} \quad x > L \\ 0, & \text{for} \quad x \le L. \end{cases} \tag{13.5}$$

In other words, $\Phi_L(x) = \Phi(x) \cdot I\{x > L\}$, where $I$ denotes the indicator function.

For further use we note that the pricing functional $F(t, S;\Phi)$ is linear in the $\Phi$-argument, and that the "chopping" operation above is also linear.

**Lemma 13.7** *For all reals* $\alpha$ *and* $\beta$, *and all functions* $\Phi$ *and* $\Psi$, *we have*

$$\begin{aligned} F(t,s;\alpha\Phi + \beta\Psi) &= \alpha F(t,s;\Phi) + \beta F(t,s;\Psi), \\ (\alpha\Phi + \beta\Psi)_L &= \alpha\Phi_L + \beta\Psi_L. \end{aligned}$$

**Proof** For $F$ the linearity follows immediately from the risk neutral valuation formula together with the linearity of the expectation operator. The linearity of the chopping operation is obvious.

Our main result is the following theorem, which tells us that the pricing problem for the down-and-out version of the contract $\Phi$ is essentially reduced to that of pricing the nonbarrier claim $\Phi_L$. Thus, if we can price a standard (nonbarrier) claim with contract function $\Phi_L$ then we can also price the down-and-out version of the contract $\Phi$.

**Theorem 13.8 (Pricing down-and-out contracts)** *Consider a fixed T-claim* $Z = \Phi(S(T))$. *Then the pricing function, denoted by* $F_{LO}$, *of the corresponding down-and-out contract* $Z_{LO}$ *is given, for* $S > L$, *by*

$$F_{LO}(t,s;\Phi) = F(t,s;\Phi_L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; \Phi_L\right). \tag{13.6}$$

*Here we have used the notation*

$$\tilde{r} = r - \frac{1}{2}\sigma^2.$$

**Proof** Without loss of generality we may set $t = 0$ in (13.6). Assume then that $S(0) = s > L$, and recall that $S_L$ denotes the process $S$ with (possible) absorption at $L$. Using risk neutral valuation we have

$$\begin{aligned} F_{LO}(0,s;\Phi) &= e^{-rT}E^Q_{0,s}[Z_{LO}] = e^{-rT}E^Q_{0,s}\left[\Phi(S(T)) \cdot I\{\inf_{0\le t\le T} S(t) > L\}\right] \\ &= e^{-rT}E^Q_{0,s}\left[\Phi_L(S_L(T)) \cdot I\{\inf_{0\le t\le T} S(t) > L\}\right] = e^{-rT}E^Q_{0,s}[\Phi_L(S_L(T))]. \end{aligned}$$

It remains to compute the last expectation, and we have

$$E^Q_{0,s}[\Phi_L(S_L(T))] = \int_L^\infty \Phi_L(x)h(x)dx.$$

where $h$ is the density function for the stochastic variable $S_L(T)$.

From standard theory we have

$$S(T) = \exp\{\ln s + \tilde{r}T + \sigma W(T)\} = e^{X(T)},$$

where the process $X$ is defined by

$$\begin{aligned} dx(t) &= \tilde{r}dt + \sigma dW(t), \\ X(0) &= \ln s. \end{aligned}$$

Thus we have

$$S_L(t) = \exp\{X_{\ln L}(t)\},$$

so we may write

$$E_{0,s}^{Q}[\Phi_L(S_L(T))] = \int_{\ln L}^{\infty} \Phi_L(e^x) f(x)dx,$$

where $f$ is the density of the stochastic variable $X_{\ln L}(T)$. This density is, however, given by Proposition 13.3 as

$$\begin{aligned} f(x) &= \varphi\left(x; \tilde{r}T + \ln s, \sigma\sqrt{T}\right) \\ &\quad -\exp\left\{-\frac{2\tilde{r}(\ln s - \ln L)}{\sigma^2}\right\}\varphi\left(x; \tilde{r}T - \ln s + 2\ln L, \sigma\sqrt{T}\right) \\ &= \varphi\left(x, \tilde{r}T + \ln s, \sigma\sqrt{T}\right) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\varphi\left(x, \tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} &E_{0,s}^{Q}[\Phi_L(S_L(T))] = \int_{\ln L}^{\infty} \Phi_L(e^x) f(x)dx \\ &= \int_{\ln L}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln s, \sigma\sqrt{T}\right)dx \\ &- \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{\ln L}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right)dx \\ &= \int_{-\infty}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln s, \sigma\sqrt{T}\right)dx \\ &- \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \int_{-\infty}^{\infty} \Phi_L(e^x)\varphi\left(x; \tilde{r}T + \ln\left(\frac{L^2}{s}\right), \sigma\sqrt{T}\right)dx. \end{aligned}$$

Inspecting the last two lines we see that the density in the first integral is the density of $X(T)$ under the usual martingale measure $Q$, given the starting value $S(0) = s$. The density in the second integral is, in the same way, the density (under $Q$) of $X(T)$, given the starting point $S(0) = L^2/s$. Thus we have

$$E_{0,s}^{Q}[\Phi_L(S_L(T))] = E_{0,s}^{Q}[\Phi_L(S(T))] - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} E_{0,\frac{L^2}{s}}^{Q}[\Phi_L(S(T))]$$

which gives us the result.

We again emphasize the point of this result.

**The problem of computing the price for a down-and-out claim reduces to the standard problem of computing the price of an ordinary (related) claim without a barrier.**

For future use we also note the fact that down-and-out pricing is a linear operation.

**Corollary 13.9** *For any contract functions* $\Phi$ *and* $\Psi$, *and for any real numbers* $\alpha$ *and* $\beta$, *the following relation holds.*

$$F_{LO}(t,s;\alpha\Phi + \beta\Psi) = \alpha F_{LO}(t,s;\Phi) + \beta F_{LO}(t,s;\Psi).$$

**Proof** The result follows immediately from Theorem 13.8 together with the linearity of the ordinary pricing functional $F$ and the linearity of the chopping operation.

## 13.2.2 Up-And-Out Contracts

We again consider a fixed $T$-contract of the form $Z = \Phi(S(T))$, and we now describe the up-and-out version of $Z$. This is the contract which at the time of delivery, $T$, will pay $Z$ if the underlying price process during the entire contract period has stayed below the barrier $L$. If, at some time during the contract period, the price process exceeds $L$, then the contract is worthless. In formal terms this reads as follows.

**Definition 13.10** *Take as given the T-contract* $Z = \Phi(S(T))$. *Then the T-contract* $Z^{LO}$ *is defined by*

$$Z^{LO} = \begin{cases} \Phi(S(T)), & \text{if} \quad S(t) < L \quad \text{for all} \quad t \in [0,T] \\ 0, & \text{if} \quad S(t) \geq L \quad \text{for some} \quad t \in [0,T]. \end{cases} \tag{13.7}$$

*The pricing functional for* $Z^{LO}$ *is denoted by* $F^{LO}(t, s;\Phi)$, *or according to our earlier notational convention, by* $\mathbf{\Phi}^{LO}(t, s)$.

$L$ as a **super** script indicates an "up"-type contract, whereas the superscript $O$ indicates that the contract is an "out" contract. As in the previous sections we will relate the up-and-out contract to an associated standard contract. To this end we need to define, for a fixed contract function $\Phi$, the function $\Phi^L$, which is the function $\Phi$ "chopped off" above $L$.

**Definition 13.11** *For a fixed function* $\Phi$ *the function* $\Phi^L$ *is defined by*

$$\Phi^L(x) = \begin{cases} \Phi(x), & \text{for} \quad x < L \\ 0, & \text{for} \quad x \geq L. \end{cases} \tag{13.8}$$

*In other words,* $\Phi^L(x) = \Phi(x) \cdot I\{x < L\}$.

The main result of this section is the following theorem, which is parallel to Theorem 13.8. The proof is almost identical.

**Theorem 13.12 (Pricing up-and-out contracts)** *Consider a fixed T-claim* $Z = \Phi(S(T))$ *. Then the pricing function,* $F^{LO}$*, of the corresponding up-and-out contract* $Z^{LO}$ *is given, for* $S < L$*, by*

$$F^{LO}(t,s,\Phi) = F(t,s,\Phi^{L}) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s},\Phi^{L}\right)$$

*where we have used the notation*

$$\tilde{r} = r - \frac{1}{2}\sigma^2. \tag{13.9}$$

## 13.2.3 Examples

In this section we will use Theorems 13.8 and 13.12, together with the linearity lemma 13.7, to give a systematic account of the pricing of a fairly wide class of barrier derivatives, including barrier call and put options. Let us define the following standard contracts, which will be the basic building blocks in the sequel.

**Definition 13.13** *Fix a delivery time T. For fixed parameters K and L define the claims ST, BO, H and C by*

$$ST(x) = x, \quad \forall x$$

$$BO(x) = 1, \quad \forall x \tag{13.10}$$

$$H(x;L) = \begin{cases} 1, & \text{if} \quad x > L \\ 0, & \text{if} \quad x \le L \end{cases} \tag{13.11}$$

$$C(x;K) = \max\quad[x-K,0]. \tag{13.12}$$

(13.13)

The contract $ST$ ($ST$ for "stock") thus gives the owner (the price of) one unit of the underlying stock at delivery time $T$, whereas $BO$ is an ordinary zero coupon bond paying one at maturity $T$. The $H$-contract ($H$ stands for the Heaviside function) gives the owner one if the value of the underlying stock exceeds $L$ at delivery time $T$, otherwise nothing is paid out. The $C$-claim is of course the ordinary European call with strike price $K$. We note in passing that $H(x;L) = H_L(x)$.

We now list the pricing functions for the standard contracts above. The value of $ST$ at time $t$ is of course equal to the value of the underlying stock at the same time, whereas the value of $BO$ at $t$ is $e^{-r(T-t)}$. The value of $C$ is given by the Black–Scholes formula, and the value of $H$ is easily calculated by using risk neutral valuation. Thus we have the following result.

**Lemma 13.14** *The contracts (13.10)–(13.13) with delivery time T are priced at time t as follows (with the pricing function in bold).*

$$\begin{aligned} \mathbf{ST(t,s)} &= \mathbf{s}, \\ \mathbf{BO(t,s)} &= \mathbf{e^{-r(T-t)}}, \end{aligned}$$

$$\mathbf{H}(t,s,L) = e^{-r(T-t)} N\left[\frac{\tilde{r}(T-t) + \ln\left(\frac{s}{L}\right)}{\sigma\sqrt{T-t}}\right],$$
$$\mathbf{C}(t,s,K) = c(t,s,K),$$

*where* $c(t, s; K)$ *is the usual Black–Scholes formula.*

We may now put this machinery to some use and start with the simple case of valuing a down-and-out contract on a bond. This contract will thus pay out 1 dollar at time $T$ if the stock price is above the level $L$ during the entire contract period, and nothing if the stock price at some time during the period is below or equal to $L$.

A direct application of Theorem 13.8 gives us the formula

$$F_{LO}(t,s,BO) = F(t,s,BO_L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s};BO_L\right).$$

Obviously we have $BO_L(x) = H(x;L)$ for all $x$ so we have the following result.

**Lemma 13.15** *The down-and-out bond with barrier* $L$ *is priced, for* $s > L$, *by the formula*

$$\mathbf{BO}_{LO}(t,s) = \mathbf{H}(t,s,L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{H}\left(t,\frac{L^2}{s};L\right), \tag{13.14}$$

*where* $\mathbf{H}(t, s; L)$ *is given by Lemma 13.14.*

We continue by pricing a down-and-out contract on the stock itself (no option is involved). Thus we want to compute $F_{LO}(t, s; ST)$ and Theorem 13.8 gives us

$$F_{LO}(t,s,ST) = F(t,s,ST_L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s};ST_L\right). \tag{13.15}$$

A quick look at a figure gives us the relation

$$ST_L(x) = L \cdot H(x;L) + C(x;L).$$

Substituting this into (13.15) and using linearity (Lemma 13.7) we get

$$\begin{aligned}
\mathbf{ST}_{LO}(t,s) &= F_{LO}(t,s,ST) \\
&= F(t,s,LH(*,L) + C(*,L)) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s};LH(*;L) + C(*;L)\right) \\
&= L \cdot F(t,s,H(*;L)) - L \cdot \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s};H(*;L)\right) \\
&\quad + F(t,s,C(*,L)) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s},C(*,L)\right)
\end{aligned}$$

Summarizing we have the following.

**Lemma 13.16** *The down-and-out contract on the underlying stock is given by*

$$\begin{aligned}\mathbf{ST}_{LO}(t,s) &= L\cdot\mathbf{H}(t,s,L) - L\cdot\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\mathbf{H}\left(t,\frac{L^2}{s};L\right)\\ &+\mathbf{C}(t,s,L) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\mathbf{C}\left(t,\frac{L^2}{s},L\right),\end{aligned}$$

*where* **H** *and* **C** *are given by Lemma 13.14.*

We now turn to a more interesting example—a down-and-out European call with strike price *K*. From the main proposition we immediately have

$$\begin{aligned}F_{LO}(t,s,C(*,K)) &= F(t,s,C_L(*,K))\\ &-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};C_L(*,K)\right),\end{aligned} \tag{13.16}$$

and we have to treat the two cases $L < K$ and $L > K$ separately. The result is as follows.

**Proposition 13.17 (Down-and-out call)** *The down-and-out European call option is priced as follows.*

*For* $L < K$*:*

$$\mathbf{C}_{LO}(t,s,K) = \mathbf{C}(t,s,K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\mathbf{C}\left(t,\frac{L^2}{s};K\right). \tag{13.17}$$

*For* $L > K$*:*

$$\begin{aligned}\mathbf{C}_{LO}(t,s,K) &= \mathbf{C}(t,s,K) + (L-K)\mathbf{H}(t,s,L)\\ &-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\left\{\mathbf{C}\left(t,\frac{L^2}{s};K\right) + (L-K)\mathbf{H}\left(t,\frac{L^2}{s};L\right)\right\}\end{aligned} \tag{13.18}$$

**Proof** For $L < K$ it is easily seen (draw a figure) that $C_L(x, K) = C(x, K)$, so from (13.16) we get

$$\begin{aligned}\mathbf{C}_{LO}(t,s,K) &= F(t,s,C(*,K))\\ &-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}F\left(t,\frac{L^2}{s};C(*,K)\right),\end{aligned}$$

which proves (13.17).

For $L > K$ the situation is slightly more complicated. Another figure shows that

$$C_L(x;K) = C(x;L) + (L-K)H(x;L).$$

Putting this relation into (13.16), and using the linear property of pricing, we get (13.18).

As we have seen, almost all results are fairly easy consequences of the linearity of the pricing functional. In Section 8.1 we used this linearity to prove the standard put–call parity relation for standard European options, and we can now derive the put–call parity result for down-and-out options.

Drawing a figure we see that $P(x;K) = K - x + C(x;K)$, so, in terms of the standard contracts, we have

$$P(x;K) = K \cdot BO(x) - ST(x) + C(x;K).$$

Using Corollary 13.9 we immediately have the following result. Note that when $L = 0$ we have the usual put–call parity.

**Proposition 13.18 (Put–call parity)** *The down-and-out put price* $\mathbf{P}_{LO}$, *and call price* $\mathbf{C}_{LO}$, *are related by the formula*

$$\mathbf{P}_{LO}(t,s;K) = K \cdot \mathbf{B}_{LO}(t,s) - \mathbf{ST}_{LO}(t,s) + \mathbf{C}_{LO}(t,s;K). \tag{13.19}$$

*Here* $\mathbf{B}_{LO}$ *and* $\mathbf{ST}_{LO}$ *are given by Lemmas13.15and13.16, whereas* $\mathbf{C}_{LO}$ *is given by Proposition13.17.*

We end this section by computing the price of a European up-and-out put option with barrier $L$ and strike price $K$.

**Proposition 13.19 (Up-and-out call)** *The price of an up-and-out European call option is given by the following formulas.*

*If* $L>K$ *then, for* $s < L$:

$$\mathbf{P}^{LO}(t,s,K) = \mathbf{P}(t,s,K) - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{P}\left(t,\frac{L^2}{s};K\right). \tag{13.20}$$

*If* $L>K$, *then for* $s < L$:

$$\begin{aligned} \mathbf{P}^{LO}(t,s,K) &= \mathbf{P}(t,s,L) - (K-L)\mathbf{H}(t,s,L) \\ &- \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left\{ \mathbf{P}\left(t,\frac{L^2}{s};L\right) - (K-L)\mathbf{H}\left(t,\frac{L^2}{s};L\right) \right\} \\ &+ \left\{ 1 - \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \right\} (K-L)e^{-r(T-t)} \end{aligned} \tag{13.21}$$

**Proof** If $L > K$ then $P^L(s;K) = P(s;K)$, and then (13.20) follows immediately from Proposition 13.12.

If $L < K$ then it is easily seen that

$$P^L(x) = P(x;L) + (K-L) \cdot BO(x) - (K-L) \cdot H(x;L).$$

Linearity and Proposition 13.12 give us (13.21).

# 13.3 In Contracts

In this section we study contracts which will **start** to exist if and only if the price of the underlying stock hits a prespecified barrier level at some time during the contract period. We thus fix a standard $T$-claim of the form $\mathcal{Z} = \Phi(S(T))$, and we also fix a barrier $L$. We start by studying the "down-and-in" version of $\mathcal{Z}$, which is defined as follows.

- If the stock price stays above the barrier $L$ during the entire contract period, then nothing is paid to the holder of the contract.
- If the stock price, at some time before the delivery time $T$, hits the barrier $L$, then the amount $\mathcal{Z}$ is paid to the holder of the contract.

We will write the down-and-in version of $\mathcal{Z}$ as $\mathcal{Z}_{LI}$, and the formal definition is as follows.

**Definition 13.20** *Take as given the T-contract* $\mathcal{Z} = \Phi(S(T))$. *Then the T-contract* $\mathcal{Z}_{LI}$ *is defined by*

$$\mathcal{Z}_{LI} = \begin{cases} 0, & if \quad S(t) > L \quad for \quad all \quad t \in [0,T] \\ \Phi(S(T)), & if \quad S(t) \le L \quad for \quad some \quad t \in [0,T]. \end{cases} \tag{13.22}$$

*The pricing function for* $\mathcal{Z}_{LI}$ *is denoted by* $F_{LI}(t, s; \Phi)$, *or sometimes by* $\mathbf{\Phi}_{LI}(t, s)$.

Concerning the notation, $L$ as a **sub** script indicates a "down" contract, whereas the subscript $I$ denotes an "in" contract. Pricing a down-and-in contract turns out to be fairly easy, since we can in fact price it in terms of the corresponding down-and-out contract.

**Lemma 13.21** [In–out parity]

$$F_{LI}(t,s;\Phi) = F(t,s;\Phi) - F_{LO}(t,s;\Phi), \quad \forall s.$$

**Proof** If, at time $t$, you have a portfolio consisting of a down-and-out version of $\mathcal{Z}$ as well as a down-and-in version of $\mathcal{Z}$ (with the same barrier $L$) then obviously you will receive exactly $\mathcal{Z}$ at time $T$. We thus have

$$F(t,s;\Phi) = F_{LI}(t,s;\Phi) + F_{LO}(t,s;\Phi).$$

We can now formulate the basic result.

**Proposition 13.22 (Pricing down-and-in contracts)** *Consider a fixed T-contract* $\mathcal{Z} = \Phi(S(T))$. *Then the price of the corresponding down-and-in contract* $\mathcal{Z}_{LI}$ *is given by*

$$F_{LI}(t,s;\Phi) = F(t,s;\Phi^L) + \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t, \frac{L^2}{s}; \Phi_L\right).$$

**Proof** From the equality $\Phi = \Phi_L + \Phi^L$ we have

$$F(t,s;\Phi) = F(t,s;\Phi_L) + F(t,s;\Phi^L).$$

Now use this formula, the lemma above, and Theorem 13.8.

The treatment of "up-and-in" contracts is of course parallel to down-and-in contracts, so we only give the basic definitions and results. We denote the up-and-in version of $\mathcal{Z}$ by $\mathcal{Z}^{LI}$, and the definition of $\mathcal{Z}^{LI}$ is as follows.

- If the stock price stays below the barrier $L$ during the entire contract period, then nothing is paid to the holder of the contract.
- If the stock price, at some time before the delivery time $T$, hits the barrier $L$, then the amount $\mathcal{Z}$ is paid to the holder of the contract.

Corresponding to Lemma 13.21, we have

$$F^{LI}(t,s;\Phi) = F(t,s;\Phi) - F^{LO}(t,s;\Phi), \quad \forall s, \tag{13.23}$$

and from this relation, together with the pricing formula for up-and-out contracts, we have an easy valuation formula.

**Proposition 13.23 (Pricing up-and-in contracts)** *Consider a fixed T-contract* $\mathcal{Z} = \Phi(S(T))$. *Then the price of the corresponding up-and-in contract* $\mathcal{Z}^{LI}$ *is given by*

$$F^{LI}(t,s;\Phi) = F(t,s;\Phi_L) + \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} F\left(t,\frac{L^2}{s};\Phi^L\right)$$

We end this section by giving, as an example, the pricing formula for a down-and-in European call with strike price $K$.

**Proposition 13.24 (Down-and-in European call)** *For $s > L$ the down-and-in European call option is priced as follows.*

*For $L < K$:*

$$\mathbf{C}_{LI}(t,s;K) = \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathbf{C}\left(t,\frac{L^2}{s};K\right).$$

*For $L > K$:*

$$\begin{aligned}\mathbf{C}_{LI}(t,s;K) &= \left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} \left\{\mathbf{C}\left(t,\frac{L^2}{s};K\right) + (L-K)\mathbf{H}\left(t,\frac{L^2}{s};L\right)\right\} \\ &\quad -(L-K)\mathbf{H}(t,s;L).\end{aligned}$$

## 13.4 Ladders

Let us take as given

- A finite increasing sequence of real numbers

$$0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N.$$

  This sequence will be denoted by α.
- Another finite increasing sequence of real numbers

$$0 = \beta_0 < \beta_1 < \ldots < \beta_N.$$

  This sequence will be denoted by β.

Note that the number $N$ is the same in both sequences. The interval $[\alpha_n, \alpha_{n+1})$ will play an important role in the sequel, and we denote it by $D_n$, with $D_N$ defined as $D_N = [\alpha_N, \infty)$. For a fixed delivery time $T$ we will now consider a new type of contract, called the "(α,β)-ladder", which is defined as follows.

**Definition 13.25** *The* (α,β)**-ladder** *with delivery time T is a T-claim* $Z$*, described by*

$$Z = \sum_{n=0}^{N} \beta_n \cdot I\{\sup_{t \le T} S(t) \in D_n\}. \tag{13.24}$$

In other words, if the realized maximum of the underlying stock during the contract period falls within the interval $D_n$, then the payout at $T$ is $\beta_n$. A typical ladder used in practice is the **forward ladder call** with strike price $K$. For this contract α is exogenously specified, and β is then defined as

$$\beta_n = \max\,[\alpha_n - K, 0]. \tag{13.25}$$

The α-sequence in this case acts as a sequence of barriers, and the ladder call allows you to buy (at time $T$) the underlying asset at the strike price $K$, while selling it (at $T$) at the highest barrier achieved by the stock price during the contract period. The ladder call is intimately connected to the lookback forward call (see the next section), to which it will converge as the α-partition is made finer.

The general (α, β)-ladder is fairly easy to value analytically, although the actual expressions may look formidable. To see this let us define the following series of up-and-in contracts.

**Definition 13.26** *For a given pair* (α, β), *the series of contracts* $Z_0, \ldots, Z_N$ *is defined by*

$$\begin{aligned} Z_0 &= \beta_0 \cdot I\{\sup_{t \le T} S(t) \ge \alpha_0\}, \\ Z_n &= (\beta_n - \beta_{n-1}) \cdot I\{\sup_{t \le T} S(t) \ge \alpha_n\}, \quad n = 1, \ldots, N. \end{aligned}$$

The point of introducing the $Z_n$-contracts is that we have the following obvious relation

$$Z = \sum_{n=0}^{N} Z_n.$$

Thus a ladder is simply a sum of a series of up-and-in contracts. We see that in fact $Z_n$ is an up-and-in contract on $\beta_n - \beta_{n-1}$ bonds, with barrier $\alpha_n$. Thus we may use the results of the preceding sections to value $Z_n$. The result is as follows.

**Proposition 13.27 (Ladder pricing formula)** *Consider an* $(\alpha, \beta)$*-ladder with delivery time T. Assume that* $S(t) = s$ *and that* $M_S(t) \in D_m$. *Then the price,* $\Pi(t)$, *of the ladder is given by*

$$\Pi(t) = \beta_m + \sum_{n=m+1}^{N} \gamma_n F^{\alpha_n I}(t, s; BO),$$

*where* $\gamma_n = \beta_n - \beta_{n-1}$, *and*

$$F^{\alpha_n I}(t,s;BO) = \left(\frac{\alpha_n}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} e^{-r(T-t)} + e^{-r(T-t)} N\left[\frac{\tilde{r}(T-t) + \ln\left(\frac{s}{\alpha_n}\right)}{\sigma\sqrt{T-t}}\right] - e^{-r(T-t)}\left(\frac{\alpha_n}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}} N\left[\frac{\tilde{r}(T-t) - \ln\left(\frac{s}{\alpha_n}\right)}{\sigma\sqrt{T-t}}\right].$$

**Proof** Exercise for the reader.

## 13.5 Lookbacks

Lookback options are contracts which at the delivery time $T$ allow you to take advantage of the realized maximum or minimum of the underlying price process over the entire contract period. Typical examples are

$$\begin{array}{ll} S(T) - \min_{t \le T} S(t) & \text{lookback call} \\ \max_{t \le T} S(t) - S(T) & \text{lookback put} \\ \max\left[\max_{t \le T} S(t) - K, 0\right] & \text{forward lookback call} \\ \max\left[K - \min_{t \le T} S(t), 0\right] & \text{forward lookback put.} \end{array}$$

We will confine ourselves to give a sketch of the pricing of a lookback put; for further results see the Notes below.

From general theory, the price of the lookback put at $t = 0$ is given by

$$\begin{aligned}\Pi(0) &= e^{-rT}E^Q[\max_{t\le T} S(t) - S(T)] \\ &= e^{-rT}E^Q[\max_{t\le T} S(t)] - e^{-rT}E^Q[S(T)].\end{aligned}$$

With $S(0) = s$, the last term is easily obtained as

$$e^{-rT}E^Q[S(T)] = s,$$

and it remains to compute the term $E^Q[\max_{t\le T} S(t)]$. To this end we recall that $S(t)$ is given by

$$S(t) = \exp\{\ln s + \tilde{r}t + \sigma W(t)\} = e^{X(t)},$$

where

$$\begin{aligned}dX &= \tilde{r}dt + \sigma dW, \\ X(0) &= \ln s.\end{aligned}$$

Thus we see that

$$M_S(T) = e^{M_X(T)},$$

and the point is of course that the distribution for $M_X(T)$ is known to us from Proposition 13.4. Using this proposition we obtain the distribution function, $F$, for $M_X(T)$ as

$$F(x) = N\left(\frac{x - \ln s - \tilde{r}T}{\sigma\sqrt{T}}\right) - \exp\left\{\frac{2\tilde{r}(x - \ln s)}{\sigma^2}\right\} N\left(-\frac{x - \ln s + \tilde{r}T}{\sigma\sqrt{T}}\right),$$

for all $x \ge \ln s$. From this expression we may compute the density function $f = F'$, and then the expected value is given by

$$E^Q[\max_{t\le T} S(t)] = E^Q\left[e^{M_X(T)}\right] = \int_{\ln s}^{\infty} e^x f(x)dx.$$

After a series of elementary, but extremely tedious, partial integrations we end up with the following result.

**Proposition 13.28 (Pricing formula for lookback put)** *The price, at $t = 0$, of the lookback put is given by*

$$\Pi(0) = -sN[-d] + se^{-rT}N\left[-d + \sigma\sqrt{T}\right] + s\frac{\sigma^2}{2r}N[d] - se^{-rT}\frac{\sigma^2}{2r}N\left[-d + \sigma\sqrt{T}\right],$$

where

$$d = \frac{rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

## 13.6 Exercises

In all exercises below we assume a standard Black–Scholes model.

**Exercise 13.1** An "all-or-nothing" contract, with delivery date $T$, and strike price $K$, will pay you the amount $K$, if the price of the underlying stock exceeds the level $L$ at some time during the interval $[0, t]$. Otherwise it will pay nothing. Compute the price, at $t < T$, of the all-or-nothing contract. In order to avoid trivialities, we assume that $S(s) < L$ for all $s \leq t$.

**Exercise 13.2** Consider a binary contract, i.e. a $T$-claim of the form

$$\chi = I_{[a,b]}(S_T),$$

where as usual $I$ is the indicator function. Compute the price of the down-and-out version of the binary contract above, for all possible values of the barrier $L$.

**Exercise 13.3** Consider a general down-and-out contract, with contract function $\Phi$, as descibed in Section 13.2.1. We now modifiy the contract by adding a fixed "rebate" $A$, and the entire contract is specified as follows.

- If $S(t) > L$ for all $t \leq T$ then $\Phi\ (S(T))$ is paid to the holder.
- If $S(t) \leq L$ for some $t \leq T$ then the holder receives the fixed amount $A$.

Derive a pricing formula for this contract.

**Hint:** Use Proposition 13.4.

**Exercise 13.4** Use the exercise above to price a down-and-out European call with rebate $A$.

**Exercise 13.5** Derive a pricing formula for a down-and-out version of the $T$ contract $\chi = \Phi(S(T))$, when $S$ has a continuous dividend yield $\delta$. Specialize to the case of a European call.

## 13.7 Notes

Most of the concrete results above are standard. The general Theorem 13.8 and its extensions seem, however, to be new. For barrier options we refer to Rubinstein and Reiner (1991), and the survey in Carr (1995). Two standard papers on lookbacks are Conze and Viswanathan (1991), and Goldman *et al.* (1979). See also Musiela and Rutkowski (1997).

# 14 Stochastic Optimal Control

## 14.1 An Example

Let us consider an economic agent over a fixed time interval $[0, T]$. At time $t = 0$ the agent is endowed with initial wealth $x_0$ and his/her problem is how to allocate investments and consumption over the given time horizon. We assume that the agent's investment opportunities are the following.

- The agent can invest money in the bank at the deterministic short rate of interest $r$, i.e. he/she has access to the risk free asset $B$ with

$$dB = rBdt. \tag{14.1}$$

- The agent can invest in a risky asset with price process $S_t$, where we assume that the $S$-dynamics are given by a standard Black–Scholes model

$$dS = \alpha Sdt + \sigma SdW. \tag{14.2}$$

We denote the agent's relative portfolio weights at time $t$ by $u_t^0$ (for the riskless asset), and $u_t^1$ (for the risky asset) respectively. His/her consumption rate at time $t$ is denoted by $c_t$.

We restrict the consumer's investment–consumption strategies to be self-financing, and as usual we assume that we live in a world where continuous trading and unlimited short selling is possible. If we denote the wealth of the consumer at time $t$ by $X_t$, it now follows from Lemma 5.4 that (after a slight rearrangement of terms) the $X$-dynamics are given by

$$dX_t = X_t[u_t^0 r + u_t^1 \alpha]\,dt - c_t dt + u_t^1 \sigma X_t dW_t. \tag{14.3}$$

The object of the agent is to choose a portfolio–consumption strategy in such a way as to maximize his/her total utility over $[0, T]$, and we assume that this utility is given by

$$E\left[\int_0^T F(t,c_t)dt + \Phi(X_T)\right], \tag{14.4}$$

where $F$ is the instantaneous utility function for consumption, whereas $\Phi$ is a “legacy” function which measures the utility of having some money left at the end of the period.

A natural constraint on consumption is the Condition

$$c_t \geq 0, \quad \forall t \geq 0,$$

and we also have of course the constraint

(14.5)

$$u_t^0 + u_t^1 = 1, \quad \forall t \geq 0.$$

(14.6)

Depending upon the actual situation we may be forced to impose other constraints (it may, say, be natural to demand that the consumer's wealth never becomes negative), but we will not do this at the moment.

We may now formally state the consumer's utility maximization problem as follows.

$$\max_{u^0, u^1, c} E\left[\int_0^T F(t, c_t)dt + \Phi(X_T)\right]$$

$$dX_t = X_t[u_t^0 r + u_t^1 \alpha]dt - c_t dt + u_t^1 \sigma X_t dW_t, \tag{14.7}$$

$$X_0 = x_0, \tag{14.8}$$

$$C_t \geq 0, \quad \forall t \geq 0, \tag{14.9}$$

$$u_t^0 + u_t^1 = 1, \quad \forall t \geq 0. \tag{14.10}$$

(14.11)

A problem of this kind is known as a **stochastic optimal control problem** . In this context the process $X$ is called the **state process** (or state variable), the processes $u^0$, $u^1$, $c$ are called **control processes** , and we have a number of **control constraints** . In the next sections we will study a fairly general class of stochastic optimal control problems. The method used is that of **dynamic programming** , and at the end of the chapter we will solve a version of the problem above.

# 14.2 The Formal Problem

We now go on to study a fairly general class of optimal control problems. To this end, let μ $(t, x, u)$ and σ $(t, x, u)$ be given functions of the form

$$\mu: R_+ \times R^n \times R^k \longrightarrow R^n,$$
$$\sigma: R_+ \times R^n \times R^k \longrightarrow R^{n \times d}.$$

For a given point $x_0 \in R^n$ we will consider the following **controlled** stochastic differential equation.

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t,$$

$$X_0 = x_0. \tag{14.12}$$

(14.13)

We view the $n$-dimensional process $X$ as a **state process** , which we are trying to "control" (or "steer"). We can (partly) control the state process $X$

by choosing the $k$-dimensional **control process** $u$ in a suitable way. $W$ is a $d$-dimensional Wiener process, and we must now try to give a precise mathematical meaning to the formal expressions (14.12)–(14.13).

**Remark 14.2.1** In this chapter, where we will work under a fixed measure, all Wiener processes are denoted by the letter $W$.

Our first modelling problem concerns the class of admissible control processes. In most concrete cases it is natural to require that the control process $u$ is adapted to the $X$ process. In other words, at time $t$ the value $u_t$ of the control process is only allowed to "depend" on past observed values of the state process $X$. One natural way to obtain an adapted control process is by choosing a deterministic function $g(t, x)$

$$g: R_+ \times R^n \to R^k,$$

and then defining the control process $u$ by

$$u_t = g(t, X_t).$$

Such a function $g$ is called a **feedback control law** , and in the sequel we will restrict ourselves to consider only feedback control laws. For mnemo-technical purposes we will often denote control laws by $\mathbf{u}(t, x)$, rather than $g(t, x)$, and write $u_t = \mathbf{u}(t, X_t)$. We use boldface in order to indicate that $\mathbf{u}$ is a **function** . In contrast to this we use the notation $u$ (italics) to denote the **value** of a control at a certain time. Thus $\mathbf{u}$ denotes a mapping, whereas $u$ denotes a point in $R^k$.

Suppose now that we have chosen a fixed control law $\mathbf{u}(t, x)$. Then we can insert $\mathbf{u}$ into (14.12) to obtain the standard SDE

$$dX_t = \mu(t, X_t, \mathbf{u}(t, X_t))dt + \sigma(t, X_t, \mathbf{u}(t, X_t))dW_t. \tag{14.14}$$

In most concrete cases we also have to satisfy some **control constraints** , and we model this by taking as given a fixed subset $U \subseteq R^k$ and requiring that $u_t \in U$ for each $t$. We can now define the class of **admissible control laws** .

**Definition 14.1** *A control law* $\mathbf{u}$ *is called* **admissible** *if*

- $\mathbf{u}(t, x) \in U$ *for all* $t \in R_+$ *and all* $x \in R^n$.
- *For any given initial point* $(t, x)$ *the SDE*

$$\begin{aligned} dX_s &= \mu(s, X_s, \mathbf{u}(s, X_s))ds + \sigma(s, X_s, \mathbf{u}(s, X_s))dW_s, \\ X_t &= x \end{aligned}$$

  *has a unique solution.*

*The class of admissible control laws is denoted by* $\mathcal{U}$.

For a given control law $\mathbf{u}$ , the solution process $X$ will of course depend on the initial value $x$, as well as on the chosen control law $\mathbf{u}$ . To be precise we should therefore denote the process $X$ by $X^{x,\mathbf{u}}$, but sometimes we will suppress $x$ or $\mathbf{u}$ .

We note that eqn (14.14) looks rather messy, and since we will also have to deal with the Itô formula in connection with (14.14) we need some more streamlined notation.

**Definition 14.2** *Consider eqn (14.14), and let ' denote matrix transpose.*

- *For any fixed vector $u \in R^k$, the functions $\mu^u$, $\sigma^u$ and $C^u(t, x)$ are defined by*

$$\begin{aligned} \mu^u(t,x) &= \mu(t,x,u), \\ \sigma^u(t,x) &= \sigma(t,x,u), \\ C^u(t,x) &= \sigma(t,x,u)\sigma(t,x,u)'. \end{aligned}$$

- *For any control law* **u** *, the functions $\mu^{\mathbf{u}}$ , $\sigma^{\mathbf{u}}$ , $C^{\mathbf{u}}(t, x)$ and $F^{\mathbf{u}}(t, x)$ are defined by*

$$\begin{aligned} \mu^{\mathbf{u}}(t,x) &= \mu(t,x,\mathbf{u}(t,x)), \\ \sigma^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x)), \\ C^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x))\sigma(t,x,\mathbf{u}(t,x))', \\ F^{\mathbf{u}}(t,x) &= F(t,x,\mathbf{u}(t,x)). \end{aligned}$$

- *For any fixed vector $u \in R^k$, the partial differential operator $\mathcal{A}^u$ is defined by*

$$\mathcal{A}^u = \sum_{i=1}^{n} \mu_i^u(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}^u(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

- *For any control law* **u** *, the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by*

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^{n} \mu_i^{\mathbf{u}}(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}^{\mathbf{u}}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Given a control law **u** we will sometimes write eqn (14.14) in a convenient shorthand notation as

$$dX_t^{\mathbf{u}} = \mu^{\mathbf{u}}dt + \sigma^{\mathbf{u}}dW_t. \tag{14.15}$$

For a given control law **u** with a corresponding controlled process $X^{\mathbf{u}}$ we will also often use the shorthand notation $\mathbf{u}_t$ instead of the clumsier expression $\mathbf{u}(t, X_t^{\mathbf{u}})$.

The reader should be aware of the fact that the existence assumption in the definition above is not at all an innocent one. In many cases it is natural to consider control laws which are “rapidly varying”, i.e. feedback laws **u** $(t, x)$ which are very irregular as functions of the state variable $x$. Inserting such an irregular control law into the state dynamics will easily give us a very irregular drift function $\mu(t, x, \mathbf{u}(t, x))$ (as a function of $x$), and we may find ourselves outside the nice Lipschitz situation in Proposition 4.1, thus leaving us with a highly nontrivial existence problem. The reader is referred to the literature for details.

We now go on to the objective function of the control problem, and therefore we consider as given a pair of functions

$$F : R_+ \times R^n \times R^k \longrightarrow R,$$
$$\Phi : R^n \longrightarrow R.$$

Now we define the **value function** of our problem as the function

$$J_0 : U \to R,$$

defined by

$$J_0(\mathbf{u}) = E\left[\int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t)dt + \Phi(X_T^{\mathbf{u}})\right].$$

where $X^{\mathbf{u}}$ is the solution to (14.14) with the given initial Condition $X_0 = x_0$.

Our formal problem can thus be written as that of maximizing $J_0(\mathbf{u})$ over all $\mathbf{u} \in U$, and we define the **optimal value** $\hat{J}_0$ by

$$\hat{J}_0 = \sup_{\mathbf{u} \in U} J_0(\mathbf{u}).$$

If there exists an admissible control law û with the property that

$$J_0(\hat{\mathbf{u}}) = \hat{J}_0.$$

then we say that û is an **optimal control law** for the given problem. Note that, as for any optimization problem, the optimal law may not exist. For a given concrete control problem our main objective is of course to find the optimal control law (if it exists), or at least to learn something about the qualitative behavior of the optimal law.

## 14.3 The Hamilton–Jacobi–Bellman Equation

Given an optimal control problem we have two natural questions to answer:

**(a)** Does there exist an optimal control law?
**(b)** Given that an optimal control exists, how do we find it?

In this text we will mainly be concerned with problem (b) above, and the methodology used will be that of **dynamic programming** . The main Idea is to embed our original problem into a much larger class of problems, and then to tie all these problems together with a partial differential equation (PDE) known as the Hamilton–Jacobi–Bellman equation. The control problem is then shown to be equivalent to the problem of finding a solution to the HJB equation.

We will now describe the embedding procedure, and for that purpose we choose a fixed point $t$ in time, with $0 \leq t \leq T$. We also choose a fixed point $x$ in the state space, i.e. $x \in R^n$. For this fixed pair $(t, x)$ we now define the following control problem.

**Definition 14.3** *The control problem* $\mathcal{P}(t,x)$ *is defined as the problem to maximize*

$$E_{t,x}\left[\int_t^T F(s,X_s^{\mathbf{u}},\mathbf{u}_s)ds + \Phi(X_T^{\mathbf{u}})\right],$$

*given the dynamics* (14.16)

$$dX_s^{\mathbf{u}} = \mu(s,X_s^{\mathbf{u}},\mathbf{u}(s,X_s^{\mathbf{u}}))ds + \sigma(s,X_s^{\mathbf{u}},\mathbf{u}(s,X_s^{\mathbf{u}}))dW_s,$$

$$X_t = x, \quad (14.17)$$

*and the constraints* (14.18)

$$\mathbf{u}(s,y) \in U, \ \forall (s,y) \in [t,T] \times R^n.$$

(14.19)

Observe that we use the notation *s* and *y* above because the letters *t* and *x* are already used to denote the fixed chosen point (*t*, *x*).

We note that in terms of the definition above, our original problem is the problem $\mathcal{P}(0,x_0)$. A somewhat drastic interpretation of the problem $\mathcal{P}(t,x)$ is that you have fallen asleep at time zero. Suddenly you wake up, noticing that the time now is *t* and that your state process while you were asleep has moved to the point *x*. You now try to do as well as possible under the circumstances, so you want to maximize your utility over the remaining time, given the fact that you start at time *t* in the state *x*.

We now define the **value function** and the **optimal value function** .

**Definition 14.4**

- *The* **value function**

$$J: R_+ \times R^n \times U \to R$$

*is defined by*

$$J(t,x,\mathbf{u}) = E\left[\int_t^T F(s,X_s^{\mathbf{u}},\mathbf{u}_s)ds + \Phi(X_T^{\mathbf{u}})\right]$$

*given the dynamics* (*14.17*)–(*14.18*).

- *The* **optimal value function**

$$V: R_+ \times R^n \to R$$

*is defined by*

$$V(t,x) = \sup_{\mathbf{u}\in} (t,x,\mathbf{u}).$$

Thus $V(t,x) = \sup_{\mathbf{u}\in U} J(t,x,\mathbf{u})$. is the expected utility of using the control law **u** over the time interval [*t*, *T*], given the fact that you start in state *x* at time *t*. The optimal value function gives you the optimal expected utility over [*t*, *T*] under the same initial conditions.

The main object of interest for us is the optimal value function, and we now go on to derive a PDE for $V$. It should be noted that this derivation is largely heuristic. We make some rather strong regularity assumptions, and we disregard a number of technical problems. We will comment on these problems later, but to see exactly which problems we are ignoring we now make some basic assumptions.

**Assumption 14.3.1** *We assume the following.*

1. *There exists an optimal control law* $\hat{u}$.
2. *The optimal value function* $V$ *is regular in the sense that* $V \in C^{1,2}$.
3. *A number of limiting procedures in the following arguments can be justified.*

We now go on to derive the PDE, and to this end we fix $(t, x) \in (0, T) \times R^n$. Furthermore we choose a real number $h$ (interpreted as a "small" time increment) such that $t + h < T$. We choose a fixed but arbitrary control law $\mathbf{u}$ , and define the control law $\mathbf{u}^*$ by

$$\mathbf{u}^*(s,y) = \begin{cases} \mathbf{u}(s,y), & (s,y) \in [t,t+h] \times R^n \\ \hat{\mathbf{u}}(s,y), & (s,y) \subset (t+h,T] \times R^n. \end{cases}$$

In other words, if we use $\mathbf{u}^*$ then we use the arbitrary control $\mathbf{u}$ during the time interval $[t, t + h]$, and then we switch to the optimal control law during the rest of the time period.

The whole Idea of dynamic programming actually boils down to the following procedure.

- First, given the point $(t, x)$ as above, we consider the following two strategies over the time interval $[t, T]$:
  **Strategy I.** Use the optimal law $\hat{u}$.
  **Strategy II.** Use the control law $\mathbf{u}^*$ defined above.
- We then compute the expected utilities obtained by the respective strategies.
- Finally, using the obvious fact that Strategy I by definition has to be at least as good as Strategy II, and letting $h$ tend to zero, we obtain our fundamental PDE.

We now carry out this program.

**Expected utility for strategy I:** This is trivial, since by definition the utility is the optimal one given by $J(t,x,\hat{\mathbf{u}}) = V(t,x)$.

**Expected utility for strategy II:** We divide the time interval $[t, T]$ into two parts, the intervals $[t, t + h)$ and $(t + h, T]$ respectively.

- The expected utility, using Strategy II, for the interval $[t, t + h)$ is given by

$$E_{t,x}\left[\int_t^{t+h} F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)\,ds\right].$$

- In the interval $[t + h, T]$ we observe that at time $t + h$ we will be in the (stochastic) state $X^{\mathbf{u}}_{t+h}$. Since, by definition, we will use the optimal strategy during the entire interval $[t + h, T]$ we see that the remaining expected utility at time $t + h$ is given by $V(t + h, X^{\mathbf{u}}_{t+h})$. Thus the expected utility over the interval $[t + h, T]$, conditional on the fact that at time $t$ we are in state $x$, is given by

$$E_{t,x}[V(t + h, X^{\mathbf{u}}_{t+h})]$$

Thus the total expected utility for Strategy II is

$$E_{t,x}\left[\int_t^{t+h} F(s, X^{\mathbf{u}}_s, \mathbf{u}_s)ds + V(t + h, X^{\mathbf{u}}_{t+h})\right].$$

**Comparing the strategies:** We now go on to compare the two strategies, and since by definition Strategy I is the optimal one, we must have the inequality

$$V(t,x) \geq E_{t,x}\left[\int_t^{t+h} F(s, X^{\mathbf{u}}_s, \mathbf{u}_s)ds + V(t + h, X^{\mathbf{u}}_{t+h})\right]. \tag{14.20}$$

We also note that the inequality sign is due to the fact that the arbitrarily chosen control law **u** which we use on the interval $[t, t + h]$ need not be the optimal one. In particular we have the following obvious fact.

**Remark 14.3.1** We have equality in (14.20) if and only if the control law **u** is an optimal law û. (Note that the optimal law does not have to be unique.)

Since, by assumption, $V$ is smooth we now use the Itô formula to obtain (with obvious notation)

$$\begin{aligned} V(t + h, X^{\mathbf{u}}_{t+h}) &= V(t,x) + \int_t^{t+h}\left\{\frac{\partial V}{\partial t}(s, X^{\mathbf{u}}_s) + A^{\mathbf{u}}V(s, X^{\mathbf{u}}_s)\right\}ds \\ &\quad + \int_t^{t+h} \nabla_x V(s, X^{\mathbf{u}}_s)\sigma^{\mathbf{u}} dW_s. \end{aligned} \tag{14.21}$$

If we apply the expectation operator $E_{t,x}$ to this equation, and assume enough integrability, then the stochastic integral will vanish. We can then insert the resulting equation into the inequality (14.20). The term $V(t, x)$ will cancel, leaving us with the inequality

$$E_{t,x}\left[\int_t^{t+h}\left[F(s, X^{\mathbf{u}}_s, \mathbf{u}_s) + \frac{\partial V}{\partial t}(s, X^{\mathbf{u}}_s) + \Box^{\mathbf{u}}V(s, X^{\mathbf{u}}_s)\right]ds\right] \leq 0. \tag{14.22}$$

**Going to the limit:** Now we divide by $h$, move $h$ within the expectation and let $h$ tend to zero. Assuming enough regularity to allow us to take the limit within the expectation, using the fundamental theorem of integral calculus, and recalling that $X_t = x$, we get

$$F(t,x,u) + \frac{\partial V}{\partial t}(t,x) + A^u V(t,x) \le 0, \tag{14.23}$$

where $u$ denotes the value of the law **u** evaluated at $(t, x)$, i.e. $u = \mathbf{u}\ (t, x)$. Since the control law **u** was arbitrary, this inequality will hold for all choices of $u \in U$, and we will have equality if and only if $u = \hat{u}(t, x)$. We thus have the following equation

$$\frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left(F(t,x,u) + A^u V(t,x)\right) = 0.$$

During the discussion the point $(t, x)$ was fixed, but since it was chosen as an arbitrary point we see that the equation holds in fact for all $(t, x) \in (0, T) \times R^n$. Thus we have a (nonstandard type of) PDE, and we obviously need some boundary conditions. One such Condition is easily obtained, since we obviously (why?) have $V(T, x) = \Phi\ (x)$ for all $x \in R^n$. We have now arrived at our goal, namely the Hamilton–Jacobi–Bellman equation, (often referred to as the HJB equation.)

**Theorem 14.5 (Hamilton–Jacobi–Bellman equation)** *Under Assumption 14.3.1, the following hold.*

1. *$V$ satisfies the Hamilton–Jacobi–Bellman equation*

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left(F(t,x,u) + A^u V(t,x)\right) = 0, & \forall (t,x) \in (0,T) \times R^n \\ V(T,x) = \Phi(x), & \forall x \in R^n \end{cases}$$

2. *For each $(t, x) \in [0, T] \times R^n$ the supremum in the HJB equation above is attained by $u = \hat{u}(t, x)$.*

**Remark 14.3.2** By going through the arguments above, it is easily seen that we may allow the constraint set $U$ to be time- and state-dependent. If we thus have control constraints of the form

$$u(t,x) \in U(t,x), \quad \forall t,x$$

then the HJB equation still holds with the obvious modification of the supremum part.

It is important to note that this theorem has the form of a **necessary** Condition. It says that **if** $V$ is the optimal value function, and **if** $\hat{u}$is the optimal control, **then** $V$ satisfies the HJB equation, and $\hat{u}(t, x)$ realizes the supremum in the equation. We also note that Assumption 14.3.1 is an *ad hoc* assumption. One would prefer to have conditions in terms of the initial data μ, σ, $F$ and Φ which would guarantee that Assumption 14.3.1 is satisfied. This can in fact be done, but at a fairly high price in terms of technical complexity. The reader is referred to the specialist literature.

A gratifying, and perhaps surprising, fact is that the HJB equation also acts as a **sufficient** Condition for the optimal control problem. This result is known as the **verification theorem** for dynamic programming, and we will use it repeatedly below. Note that, as opposed to the necessary conditions above, the verification theorem is very easy to prove rigorously.

**Theorem 14.6 (Verification theorem)** *Suppose that we have two functions* $H(t, x)$ *and* $g(t, x)$, *such that*

- *H is sufficiently integrable (see Remark (14.3.4) below), and solves the HJB equation*

$$\begin{cases} \frac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + A^u H(t,x)\} = 0, & \forall (t,x) \in (0,T) \times R^n \\ H(T,x) = \Phi(x), & \forall x \in R^n \end{cases}$$

- *The function g is an admissible control law.*
- *For each fixed* $(t, x)$, *the supremum in the expression*

$$\sup_{u \in U} \{F(t,x,u) + A^u H(t,x)\}$$

  *is attained by the choice* $u = g(t, x)$.

*Then the following hold.*

1. *The optimal value function V to the control problem is given by*

$$V(t,x) = H(t,x).$$

2. *There exists an optimal control law* $\hat{u}$, *and in fact* $\hat{u}(t, x) = g(t, x)$.

**Remark 14.3.3** Note that we have used the letter $H$ (instead of $V$) in the HJB equation above. This is because the letter $V$ by definition denotes the optimal value function.

**Proof** Assume that $H$ and $g$ are given as above. Now choose an arbitrary control law $\mathbf{u} \in U$, and fix a point $(t, x)$. We define the process $X^{\mathbf{u}}$ on the time interval $[t, T]$ as the solution to the equation

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu^{\mathbf{u}}(s, X_s^{\mathbf{u}})ds + \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}})dW_s, \\ X_t &= x. \end{aligned}$$

Inserting the process $X^{\mathbf{u}}$ into the function $H$ and using the Itô formula we obtain

$$\begin{aligned} H(T, X_T^{\mathbf{u}}) &= H(t,x) + \int_t^T \left\{ \frac{\partial H}{\partial t}(s, X_s^{\mathbf{u}}) + (A^{\mathbf{u}})(s, X_s^{\mathbf{u}}) \right\} ds \\ &+ \int_t^T \nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) dW_s. \end{aligned}$$

Since $H$ solves the HJB equation we see that

$$\frac{\partial H}{\partial t}(t,x) + F(t,x,u) + A^{u}H(t,x) \leq 0$$

for all $u \in U$, and thus we have, for each $s$ and $P$-a.s, the inequality

$$\frac{\partial H}{\partial t}(s,X_s^{\mathbf{u}}) + (A^{\mathbf{u}})(s,X_s^{\mathbf{u}}) \leq -F^{\mathbf{u}}(s,X_s^{\mathbf{u}}).$$

From the boundary Condition for the HJB equation we also have $H(T,X_T^{\mathbf{u}}) = \Phi(X_T^{\mathbf{u}})$, so we obtain the inequality

$$H(t,x) \geq \int_t^T F^{\mathbf{u}}(s,X_s^{\mathbf{u}})ds + \Phi(X_T^{\mathbf{u}}) + \int_t^T \nabla_x H(s,X_s^{\mathbf{u}})\sigma^{\mathbf{u}}dW_s.$$

Taking expectations, and assuming enough integrability, we make the stochastic integral vanish, leaving us with the inequality

$$H(t,x) \geq E_{t,x}\left[\int_t^T F^{\mathbf{u}}(s,X_s^{\mathbf{u}})ds + \Phi(X_T^{\mathbf{u}})\right] = J(t,x,\mathbf{u}).$$

Since the control law **u** was arbitrarily chosen this gives us

$$H(t,x) \geq \sup_{u \in U} J(t,x,\mathbf{u}) = V(t,x).$$

(14.24)

To obtain the reverse inequality we choose the specific control law **u** $(t, x)$ = **g** $(t, x)$. Going through the same calculations as above, and using the fact that by assumption we have

$$\frac{\partial H}{\partial t}(t,x) + F\mathbf{g}(t,x) + A^{\mathbf{g}}H(t,x) = 0,$$

we obtain the equality

$$H(t,x) = E_{t,x}\left[\int_t^T F^{g}(s,X_s^{g})ds + \Phi(X_T^{g})\right] = J(t,x,g).$$

(14.25)

On the other hand we have the trivial inequality

$$V(t,x) \geq J(t,x,\mathbf{g}).$$

(14.26)

so, using (14.24)–(14.26), we obtain

$$H(t,x) \geq V(t,x) \geq J(t,x,\mathbf{g}) = H(t,x).$$

This shows that in fact

$$H(t,x) = V(t,x) = J(t,x,\mathbf{g}).$$

which proves that $H = V$, and that **g** $s$ is the optimal control law.

**Remark 14.3.4** The assumption that $H$ is "sufficiently integrable" in the theorem above is made in order for the stochastic integral in the proof to have expected value zero. This will be the case if, for example, $H$ satisifes the Condition

$$\nabla_x H(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}}(s, X_s^{\mathbf{u}}) \in \pounds^2,$$

for all admissible control laws.

**Remark 14.3.5** Sometimes, instead of a maximization problem, we consider a minimization problem. Of course we now make the obvious definitions for the value function and the optimal value function. It is then easily seen that all the results above still hold if the expression

$$\sup_{u \in U} \{F(t,x,u) + A^u V(t,x)\}$$

in the HJB equation is replaced by the expression

$$\inf_{u \in U} \{F(t,x,u) + A^u V(t,x)\}.$$

**Remark 14.3.6** In the Verification Theorem we may allow the control constraint set $U$ to be state and time dependent, i.e. of the form $U(t, x)$.

## 14.4 Handling the HJB Equation

In this section we will describe the actual handling of the HJB equation, and in the next section we will study a classical example—the linear quadratic regulator. We thus consider our standard optimal control problem with the corresponding HJB equation:

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + A^u V(t,x)\} & = 0, \\ V(T,x) & = \Phi(x). \end{cases} \tag{14.27}$$

Schematically we now proceed as follows.

1. Consider the HJB equation as a PDE for an unknown function $V$.
2. Fix an arbitrary point $(t, x) \in [0, T] \times R^n$ and solve, for this fixed choice of $(t, x)$, the static optimization problem

   $$\max_{u \in U} [F(t,x,u) + A^u V(t,x)].$$

   Note that in this problem $u$ is the only variable, whereas $t$ and $x$ are considered to be fixed parameters. The functions $F$, $\mu$, $\sigma$ and $V$ are considered as given.
3. The optimal choice of $u$, denoted by $\hat{u}$, will of course depend on our choice of $t$ and $x$, but it will also depend on the function $V$ and its various partial derivatives (which are hiding under the sign $A^u V$). To highlight these dependencies we write $\hat{u}$ as

   $$\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, x; V). \tag{14.28}$$

4. The function $\hat{u}$ ($t$, $x$; $V$) is our candidate for the optimal control law, but since we do not know $V$ this description is incomplete. Therefore we substitute the expression for $\hat{u}$ in (14.28) into the PDE (14.27), giving us the PDE

$$\frac{\partial V}{\partial t}(t,x) + F^{\hat{\mathbf{u}}}(t,x) + A^{\hat{\mathbf{u}}}(t,x)V(t,x) = 0, \tag{14.29}$$

$$V(T,x) = \Phi(x). \tag{14.30}$$

5. Now we solve the PDE above! (See the Remark below.) Then we put the solution $V$ into expression (14.28). Using the verification theorem 14.6 we can now identify $V$ as the optimal value function, and $\hat{u}$ as the optimal control law.

**Remark 14.4.1** The hard work of dynamic programming consists in solving the highly nonlinear PDE in step 5 above. There are of course no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed. In an actual case one usually tries to **guess** a solution, i.e. we typically make an **ansatz** for $V$, parameterized by a finite number of parameters, and then we use the PDE in order to identify the parameters. The making of an ansatz is often helped by the intuitive observation that if there is an analytical solution to the problem, then it seems likely that $V$ inherits some structural properties from the boundary function $\Phi$ as well as from the instantaneous utility function $F$.

For a general problem there is thus very little hope of obtaining an analytic solution, and it is worth pointing out that many of the known solved control problems have, to some extent, been "rigged" in order to be analytically solvable.

## 14.5 The Linear Regulator

We now want to put the ideas from the previous section into action, and for this purpose we study the most well known of all control problems, namely the linear quadratic regulator problem. In this classical engineering example we wish to **minimize**

$$E\left[\int_0^T \{X_t'QX_t + u_t'Ru_t\}\,dt + X_T'HX_T\right],$$

(where ' denotes transpose) given the dynamics

$$dX_t = \{AX_t + Bu_t\}\,dt + CdW_t.$$

One interpretation of this problem is that we want to control a vehicle in such a way that it stays close to the origin (the terms $x'Qx$ and $x'Hx$) while at the same time keeping the "energy" $u'Ru$ small.

As usual $X_t \in R^n$ and $\mathbf{u}_t \in R^k$, and we impose no control constraints on $u$. The matrices $Q$, $R$, $H$, $A$, $B$ and $C$ are assumed to be known. Without loss of generality we may assume that $Q$, $R$ and $H$ are symmetric, and we assume that $R$ is positive definite (and thus invertible).

The HJB equation now becomes

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \inf_{u \in R^k} \{x'Qx + u'Ru + [\nabla_x V](t,x)[Ax + Bu]\} \\ \qquad + \frac{1}{2}\sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}(t,x)[CC']_{i,j} = 0, \\ V(T,x) = x'Hx. \end{cases}$$

For each fixed choice of $(t, x)$ we now have to solve the static unconstrained optimization problem to minimize

$$u'Ru + [\nabla_x V](t,x)[Ax + Bu].$$

Since, by assumption, $R>0$ we get the solution by setting the gradient equal to zero, thus giving us the equation

$$2u'R = -(\nabla_x V)B,$$

which gives us the optimal $u$ as

$$\hat{u} = -\frac{1}{2}R^{-1}B'(\nabla_x V)'.$$

Here we see clearly (compare point 2 in the scheme above) that in order to use this formula we need to know $V$, and we thus try to make an educated guess about the structure of $V$. From the boundary value function $x'Hx$ and the quadratic term $x'Qx$ in the instantaneous cost function it seems reasonable to assume that $V$ is a quadratic function. Consequently we make the following ansatz

$$V(t,x) = x'P(t)x + q(t),$$

where we assume that $P(t)$ is a deterministic symmetric matrix function of time, whereas $q(t)$ is a scalar deterministic function. It would of course also be natural to include a linear term of the form $L(t)x$, but it turns out that this is not necessary.

With this trial solution we have, suppressing the $t$-variable and denoting time derivatives by a dot,

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) &= x'\dot{P}x + \dot{q}, \\ \nabla_x V(t,x) &= 2x'P, \\ \nabla_{xx} V(t,x) &= 2P \end{aligned}$$

$$\hat{u} = -R^{-1}B'Px.$$

Inserting these expressions into the HJB equation we get

$$x'\dot{P}x + \dot{q} + x'Qx + x'PBR^{-1}RR^{-1}B'Px + 2x'PAx$$
$$-2x'PBR^{-1}B'Px + \sum_{i,j} P_{ij}[CC']_{ij} = 0.$$

We note that the last term above equals $tr[C'PC]$, where $tr$ denote the trace of a matrix, and furthermore we see that $2x'PAx = x'A'Px + x'PAx$ (this is just cosmetic). Collecting terms gives us

$$x'\left\{\dot{P} + Q - PBR^{-1}B'P + A'P + PA\right\}x + \dot{q} + tr[C'PC] = 0. \tag{14.31}$$

If this equation is to hold for all $x$ and all $t$ then firstly the bracket must vanish, leaving us with the matrix ODE

$$\dot{P} = PBR^{-1}B'P - A'P - PA - Q.$$

We are then left with the scalar equation

$$\dot{q} = -tr[C'PC].$$

We now need some boundary values for $P$ and $q$, but these follow immediately from the boundary conditions of the HJB equation. We thus end up with the following pairs of equations

$$\begin{cases} \dot{P} &= PBR^{-1}B'P - A'P - PA - Q, \\ P(T) &= H. \end{cases} \tag{14.32}$$

$$\begin{cases} \dot{q} &= -tr[C'PC], \\ q(T) &= 0. \end{cases} \tag{14.33}$$

The matrix equation (14.32) is known as a **Riccati equation** , and there are powerful algorithms available for solving it numerically. The equation for $q$ can then be integrated directly.

Summing up we see that the optimal value function and the optimal control law are given by the following formulas. Note that the optimal control is linear in the state variable.

$$V(t,x) = x'P(t)x + \int_t^T tr[C'P(s)C]\,ds, \tag{14.34}$$

$$\hat{\mathbf{u}}(t,x) = -R^{-1}B'P(t)x. \tag{14.35}$$

# 14.6 Optimal Consumption and Investment

## 14.6.1 A Generalization

In many concrete applications, in particular in economics, it is natural to consider an optimal control problem, where the state variable is constrained to stay within a prespecified domain. As an example it may be reasonable to demand that the wealth of an investor is never allowed to become negative. We will now generalize our class of optimal control problems to allow for such considerations.

Let us therefore consider the following controlled SDE

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \tag{14.36}$$

$$X_0 = x_0, \tag{14.37}$$

where as before we impose the control constraint $u_t \in U$. We also consider as given a fixed time interval $[0, T]$, and a fixed domain $D \subseteq [0, T] \times R^n$, and the basic Idea is that when the state process hits the boundary $\partial D$ of $D$, then the activity is at an end. It is thus natural to define the **stopping time** $\tau$ by

$$\tau = \inf \{t \geq 0 \ |(t, X_t) \in \partial D\} \wedge T,$$

where $x \wedge y = \min [x, y]$. We consider as given an instantaneous utility function $F(t, x, u)$ and a "bequest function" $\Phi(t, x)$, i.e. a mapping $\Phi : \partial D \to R$. The control problem to be considered is that of maximizing

$$E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi(\tau, X_\tau^{\mathbf{u}})\right]. \tag{14.38}$$

In order for this problem to be interesting we have to demand that $X_0 \in D$, and the interpretation is that when we hit the boundary $\partial D$, the game is over and we obtain the bequest $\Phi(\tau, X_\tau)$. We see immediately that our earlier situation corresponds to the case when $D = [0, T] \times R^n$ and when $\Phi$ is constant in the $t$-variable.

In order to analyze our present problem we may proceed as in the previous sections, introducing the value function and the optimal value function exactly as before. The only new technical problem encountered is that of considering a stochastic integral with a stochastic limit of integration. Since this will take us outside the scope of the present text we will confine ourselves to giving the results. The proofs are (modulo the technicalities mentioned above) exactly as before.

**Theorem 14.7 (HJB equation)** *Assume that*

- *The optimal value function $V$ is in $C^{1,2}$.*
- *An optimal law $\hat{\mathbf{u}}$ exists.*

*Then the following hold.*

1. *V satisifies the HJB equation*

$$\begin{cases} \frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + A^u V(t,x)\} & = 0, \quad \forall (t,x) \subset D \\ V(t,x) & = \Phi(t,x), \quad \forall (t,x) \in \partial D. \end{cases}$$

2. *For each* $(t, x) \in D$ *the supremum in the HJB equation above is attained by* $u = \hat{u}(t, x)$.

**Theorem 14.8 (Verification theorem)** *Suppose that we have two functions* $H(t, x)$ *and* $g(t, x)$, *such that*

- *H is sufficiently integrable, and solves the HJB equation*

$$\begin{cases} \frac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + A^u H(t,x)\} & = 0, \quad \forall (t,x) \subset D \\ H(t,x) & = \Phi(t,x), \quad \forall (t,x) \in \partial D. \end{cases}$$

- *The function g is an admissible control law.*
- *For each fixed* $(t, x)$, *the supremum in the expression*

$$\sup_{u \in U} \{F(t,x,u) + A^u H(t,x)\}$$

*is attained by the choice* $u = g(t, x)$.

*Then the following hold.*

1. *The optimal value function V to the control problem is given by*

$$V(t,x) = H(t,x).$$

2. *There exists an optimal control law* $\hat{u}$, *and in fact* $\hat{u}(t, x) = g(t, x)$.

## 14.6.2 Optimal Consumption

In order to illustrate the technique we will now go back to the optimal consumption problem at the beginning of the chapter. We thus consider the problem of maximizing

$$E\left[\int_0^T F(t,c_t)dt + \Phi(X_T)\right],$$

given the wealth dynamics

$$dX_t = X_t\left[u_t^0 r + u_t^1 \alpha\right]dt - c_t dt + u^1 \sigma X_t dW_t. \tag{14.39}$$

As usual we impose the control constraints

$$\begin{aligned} c_t & \geq 0, \quad \forall t \geq 0, \\ u_t^0 + u_t^1 & = 1, \quad \forall t \geq 0. \end{aligned} \tag{14.40}$$

In a control problem of this kind it is important to be aware of the fact that one may quite easily formulate a nonsensical problem. To take a simple example, suppose that we have $\Phi = 0$, and suppose that $F$ is increasing and unbounded in the $c$-variable. Then the problem above degenerates completely. It does not possess an optimal solution at all, and the reason is of course that the consumer can increase his/her utility to any given level by simply consuming an arbitrarily large amount at every $t$. The consequence of this hedonistic behavior is of course the fact that the wealth process will, with very high probability, become negative, but this is neither prohibited by the control constraints, nor punished by any bequest function.

An elegant way out of this dilemma is to choose the domain $D$ of the preceding section as $D = [0,T] \times \{x \ | x > 0\}$. With $\tau$ defined as above this means, in concrete terms, that

$$\tau = \inf\{t > 0 \ | X_t = 0\} \wedge T.$$

A natural objective function in this case is thus given by

$$E\left[\int_0^\tau F(t,c_t)dt\right], \tag{14.41}$$

which automatically ensures that when the consumer has no wealth, then all activity is terminated.

We will now analyze this problem in some detail. Firstly we notice that we can get rid of the constraint $u_t^0 + u_t^1 = 1$ by defining a new control variable $w$ as $w = u^1$, and then substituting $1-w$ for $u^0$. This gives us the state dynamics

$$dX_t = w_t[\alpha - r]X_t dt + (rX_t - c_t)dt + w\sigma X_t dW_t, \tag{14.42}$$

and the corresponding HJB equation is

$$\begin{cases} \dfrac{\partial V}{\partial t} + \sup\limits_{c \leq 0, w \in R} \left\{ F(t,c) + wx(\alpha - r)\dfrac{\partial V}{\partial x} + (rx - c)\dfrac{\partial V}{\partial x} + \dfrac{1}{2}x^2 w^2 \sigma^2 \dfrac{\partial^2 V}{\partial x^2} \right\} & = 0, \\ V(T,x) & = 0, \\ V(t,0) & = 0. \end{cases}$$

We now specialize our example to the case when $F$ is of the form

$$F(t,c) = e^{-\delta t} c^\gamma,$$

where $0 < \gamma < 1$. The economic reasoning behind this is that we now have an infinite marginal utility at $c = 0$. This will force the optimal consumption plan to be positive throughout the planning period, a fact which will facilitate the analytical treatment of the problem. In terms of Remark 14.4.1 we are thus "rigging" the problem.

The static optimization problem to be solved w.r.t. $c$ and $w$ is thus that of maximizing

$$e^{-\delta t} c^\gamma + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial xx^2},$$

and, assuming an interior solution, the first order conditions are

$$\gamma c^{\gamma-1} = e^{\delta t} V_x, \tag{14.43}$$

$$w = \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, \tag{14.44}$$

where we have used subscripts to denote partial derivatives.

We again see that in order to implement the optimal consumption–investment plan (14.43)–(14.44) we need to know the optimal value function $V$. We therefore suggest a trial solution (see Remark 14.4.1), and in view of the shape of the instantaneous utility function it is natural to try a $V$-function of the form

$$V(t,x) = e^{-\delta t} h(t) x^{\gamma}, \tag{14.45}$$

where, because of the boundary conditions, we must demand that

$$h(T) = 0. \tag{14.46}$$

Given a $V$ of this form we have (using $\cdot$ to denote the time derivative)

$$\frac{\partial V}{\partial t} = e^{-\delta t} \dot{h} x^{\gamma} - \delta e^{-\delta t} h x^{\gamma}, \tag{14.47}$$

$$\frac{\partial V}{\partial x} = \gamma e^{-\delta t} h x^{\gamma-1}, \tag{14.48}$$

$$\frac{\partial^2 V}{\partial x^2} = \gamma(\gamma - 1) e^{-\delta t} h x^{\gamma-2}. \tag{14.49}$$

Inserting these expressions into (14.43)–(14.44) we get

$$\hat{w}(t,x) = \frac{\alpha - r}{\sigma^2(1-\gamma)}, \tag{14.50}$$

$$\hat{c}(t,x) = x h(t)^{-1/(1-\gamma)}. \tag{14.51}$$

This looks very promising: we see that the candidate optimal portfolio is constant and that the candidate optimal consumption rule is linear in the wealth variable. In order to use the verification theorem we now want to show that a $V$-function of the form (14.45) actually solves the HJB equation. We therefore substitute the expressions (14.47)–(14.51) into the HJB equation. This gives us the equation

$$x^{\gamma} \left\{ \dot{h}(t) + A h(t) + B h(t)^{-\gamma/(1-\gamma)} \right\} = 0,$$

where the constants $A$ and $B$ are given by

$$\begin{aligned} A &= \frac{\gamma(\alpha - r)^2}{\sigma^2(1-\gamma)} + r\gamma - \frac{1}{2} \frac{\gamma(\alpha - r)^2}{\sigma^2(1-\gamma)} - \delta \\ B &= 1 - \gamma. \end{aligned}$$

If this equation is to hold for all $x$ and all $t$, then we see that $h$ must solve the ODE

$$\dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} = 0, \tag{14.52}$$

$$h(T) = 0. \tag{14.53}$$

An equation of this kind is known as a **Bernoulli equation** , and it can be solved explicitly (see the exercises).

Summing up, we have shown that if we define $V$ as in (14.45) with $h$ defined as the solution to (14.52)–(14.53), and if we define $\hat{w}$ and $\hat{c}$ by (14.50)–(14.51), then $V$ satisfies the HJB equation, and $\hat{w}$, $\hat{c}$ attain the supremum in the equation. The verification theorem then tells us that we have indeed found the optimal solution.

## 14.7 The Mutual Fund Theorems

In this section we will briefly go through the "Merton mutual fund theorems", originally presented in Merton (1971).

### 14.7.1 The Case With No Risk Free Asset

We consider a financial market with $n$ asset prices $S_1, \ldots, S_n$. To start with we do not assume the existence of a risk free asset, and we assume that the price vector process $S(t)$ has the following dynamics under the objective measure $P$.

$$dS = D(S)\alpha dt + D(S)\sigma dW. \tag{14.54}$$

Here $W$ is a $k$-dimensional standard Wiener process, $\alpha$ is an $n$-vector, $\sigma$ is an $n \times k$ matrix, and $D(S)$ is the diagonal matrix

$$D(S) = diag[S_1, \ldots, S_n].$$

In more pedestrian terms this means that

$$dS_i = S_i\alpha_i dt + S_i\sigma_i dW,$$

where $\sigma_i$ is the $i$th row of the matrix $\sigma$.

We denote the investment strategy (relative portfolio) by $w$, and the consumption plan by $c$. If the pair $(w, c)$ is self-financing, then it follows from the $S$-dynamics above, and from Lemma 5.4, that the dynamics of the wealth process $X$ are given by

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW. \tag{14.55}$$

We also take as given an instantaneous utility function $F(t, c)$, and we basically want to maximize

$$E\left[\int_0^T F(t,c_t)dt\right]$$

where $T$ is some given time horizon. In order not to formulate a degenerate problem we also impose the Condition that wealth is not allowed to become negative, and as before this is dealt with by introducing the stopping time

$$\tau = \inf\{t > 0 \mid X_t = 0\} \wedge T.$$

Our formal problem is then that of maximizing

$$E\left[\int_0^\tau F(t,c_t)dt\right]$$

given the dynamics (14.54)–(14.55), and subject to the control constraints

$$\sum_1^n w_i = 1, \tag{14.56}$$

$$c \geq 0. \tag{14.57}$$

Instead of (14.56) it is convenient to write

$$e'w = 1,$$

where $e$ is the vector in $R^n$ which has the number 1 in all components, i.e. $e' = (1, \ldots, 1)$.

The HJB equation for this problem now becomes

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x,s) + \sup\limits_{e'w=1,c\geq 0} \{F(t,c) + \mathcal{A}^{c,w}V(t,x,s)\} &= 0, \\ V(T,x,s) &= 0, \\ V(t,0,s) &= 0. \end{cases}$$

In the general case, when the parameters α and σ are allowed to be functions of the price vector process $S$, the term $\mathcal{A}^{c,w}V(t,x,s)$ turns out to be rather forbidding (see Merton's original paper). It will in fact involve partial derivatives to the second order with respect to all the variables $x, s_1, \ldots, s_n$.

If, however, we assume that α and σ are deterministic and constant over time, then we see by inspection that the wealth process $X$ is a Markov process, and since the price processes do not appear, neither in the objective function nor in the definition of the stopping time, we draw the conclusion that in this case $X$ itself will act as the state process, and we may forget about the underlying $S$-process completely.

Under these assumptions we may thus write the optimal value function as $V(t, x)$, with no $s$-dependence, and after some easy calculations the term $A^{c,w}V$ turns out to be

$$A^{c,w}V = xw'\alpha\frac{\partial V}{\partial x} - c\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w'\Sigma w\frac{\partial^2 V}{\partial x^2},$$

where the matrix $\Sigma$ is given by

$$\Sigma = \sigma\sigma'.$$

We now summarize our assumptions.

**Assumption 14.7.1** *We assume that*

- *The vector* $\alpha$ *is constant and deterministic.*
- *The matrix* $\sigma$ *is constant and deterministic.*
- *The matrix* $\sigma$ *has rank* $n$, *and in particular the matrix* $\Sigma = \sigma\,\sigma'$ *is positive definite and invertible.*

We note that, in terms of contingent claims analysis, the last assumption means that the market is complete. Denoting partial derivatives by subscripts we now have the following HJB equation

$$\begin{cases} V_t(t,x) + \sup_{w'e=1, c\geq 0}\left\{F(t,c) + (xw'\alpha - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x)\right\} &= 0, \\ V(T,x) &= 0, \\ V(t,0) &= 0. \end{cases}$$

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

$$L = F(t,c) + (xw'\alpha - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x) + \lambda(1 - w'e).$$

Assuming the problem to be regular enough for an interior solution we see that the first order Condition for $c$ is

$$\frac{\partial F}{\partial c}(t,c) = V_x(t,x).$$

The first order Condition for $w$ is

$$x\alpha' V_x + x^2 V_{xx} w'\Sigma = \lambda e',$$

so we can solve for $w$ in order to obtain

$$\hat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}}e - \frac{xV_x}{x^2 V_{xx}}\alpha\right]. \tag{14.58}$$

Using the relation $e'w = 1$ this gives $\lambda$ as

$$\lambda = \frac{x^2V_{xx} + xV_x e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

and inserting this into (14.58) gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \frac{V_x}{xV_{xx}}\Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right]. \tag{14.59}$$

To see more clearly what is going on we can write this expression as

$$\hat{w}(t) = g + Y(t)h, \tag{14.60}$$

where the fixed vectors $g$ and $h$ are given by

$$g = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \tag{14.61}$$

$$h = \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \tag{14.62}$$

whereas $Y$ is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}. \tag{14.63}$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" $\Delta$, where

$$\Delta = \{w \in R^n \mid e'w = 1\}.$$

We now make the obvious geometric observation that if we fix two points on the optimal portfolio line, say the points $w^a = g + ah$ and $w^b = g + bh$, then any point $w$ on the line can be written as an affine combination of the basis points $w^a$ and $w^b$. An easy calculation shows that if $w^s = g + sh$ then we can write

$$w^s = \mu w^a + (1-\mu)w^b,$$

where

$$\mu = \frac{s-b}{a-b}.$$

The point of all this is that we now have an interesting economic interpretation of the optimality results above. Let us thus fix $w^a$ and $w^b$ as above on the optimal portfolio line. Since these points are in the portfolio plane $\Delta$ we can interpret them as the relative portfolios of two fixed mutual funds. We may then write (14.60) as

$$\hat{w}(t) = \mu(t)w^a + (1 - \mu(t))w^b, \tag{14.64}$$

with

$$\mu(t) = \frac{Y(t) - b}{a - b}.$$

Thus we see that the optimal portfolio $\hat{w}$ can be obtained as a"super portfolio" where we allocate resources between two fixed mutual funds.

**Theorem 14.9 (Mutual fund theorem)** *Assume that the problem is regular enough to allow for an interior solution. Then there exists a one-dimensional parameterized family of mutual funds, given by $w^s = g + sh$, where g and h are defined by (14.61)–(14.62), such that the following hold.*

1. *For each fixed s the relative portfolio $w^s$ stays fixed over time.*
2. *For any fixed choice of $a \neq b$ the optimal portfolio $\hat{w}(t)$ is, for all values of t, obtained by allocating all resources between the fixed funds $w^a$ and $w^b$, i.e.*

$$\begin{aligned} \hat{w}(t) &= \mu^a(t)w^a + \mu^b(t)w^b, \\ \mu^a(t) + \mu^b(t) &= 1. \end{aligned}$$

3. *The relative proportions $(\mu^a, \mu^b)$ of the portfolio wealth allocated to $w^a$ and $w^b$ respectively are given by*

$$\begin{aligned} \mu^a(t) &= \frac{Y(t) - b}{a - b}, \\ \mu^b(t) &= \frac{a - Y(t)}{a - b}, \end{aligned}$$

*where Y is given by (14.63).*

## 14.7.2 The Case With a Risk Free Asset

Again we consider the model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t), \tag{14.65}$$

with the same assumptions as in the preceding section. We now also take as given the standard risk free asset B with dynamics

$$dB = rBdt.$$

Formally we can denote this as a new asset by subscript zero, i.e. $B = S_0$, and then we can consider relative portfolios of the form $w = (w_0, w_1, \ldots, w_n)'$ where of course $\sum_0^n w_i = 1$. Since $B$ will play such a special role it will, however, be convenient to eliminate $w_0$ by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

and then use the letter $w$ to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

and we note that this truncated portfolio vector is allowed to take any value in $R^n$.

Given this notation it is easily seen that the dynamics of a self-financing portfolio are given by

$$dX = X \cdot \left\{ \sum_1^n w_i \alpha_i + \left(1 - \sum_1^n w_i\right) r \right\} dt - c dt + X \cdot w' \sigma dW.$$

That is,

$$dX = X \cdot w' (\alpha - re) dt + (rX - c) dt + X \cdot w' \sigma dW, \tag{14.66}$$

where as before $e \in R^n$ denotes the vector $(1,1,\ldots,1)'$.

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \left\{ F(t,c) + A^{c,w} V(t,x) \right\} & = 0, \\ V(T,x) & = 0, \\ V(t,0) & = 0. \end{cases}$$

where

$$A^c V = xw'(\alpha - re) V_x(t,x) + (rx - c) V_x(t,x) + \frac{1}{2} x^2 w' \Sigma w V_{xx}(t,x).$$

The first order conditions for the static optimization problem are

$$\begin{aligned} \frac{\partial F}{\partial c}(t,c) &= V_x(t,x), \\ \hat{w} &= -\frac{V_x}{x V_{xx}} \Sigma^{-1} (\alpha - re), \end{aligned}$$

and again we have a geometrically obvious economic interpretation.

**Theorem 14.10 (Mutual fund theorem)** *Given assumptions as above, the following hold.*

1. *The optimal portfolio consists of an allocation between two fixed mutual funds $w^0$ and $w^f$.*

2. *The fund $w^0$ consists only of the risk free asset.*
3. *The fund $w^f$ consists only of the risky assets, and is given by*

$$w^f = \Sigma^{-1}(\alpha - re).$$

4. *At each t the optimal relative allocation of wealth between the funds is given by*

$$\mu^f(t) = -\frac{V_x(t,X(t))}{X(t)V_{xx}(t,X(t))},$$
$$\mu^0(t) = 1 - \mu^f(t).$$

Note that this result is not a corollary of the corresponding result from the previous section. Firstly it was an essential ingredient in the previous results that the volatility matrix of the price vector was invertible. In the case with a riskless asset the volatility matrix for the entire price vector $(B, S_1, \ldots, S_n)$ is of course degenerate, since its first row (having subscript zero) is identically equal to zero. Secondly, even if one assumes the results from the previous section, i.e. that the optimal portfolio is built up from two fixed portfolios, it is not at all obvious that one of these basis portfolios can be chosen so as to consist of the risk free asset alone.

## 14.8 Exercises

**Exercise 14.1** Solve the problem of maximizing logarithmic utility

$$E\left[\int_0^T e^{-\delta t}\ln(c_t)dt + K \cdot \ln(X_T)\right],$$

given the usual wealth dynamics

$$dX_t = X_t\left[u_t^0 r + u_t^1 \alpha\right]dt - c_t dt + u^1 \sigma X_t dW_t,$$

and the usual control constraints

$$c_t \geq 0, \ \forall t \geq 0,$$
$$u_t^0 + u_t^1 = 1, \ \forall t \geq 0.$$

**Exercise 14.2** A **Bernoulli equation** is an ODE of the form

$$\dot{x}_t + A_t x_t + B_t x_t^\alpha = 0,$$

where $A$ and $B$ are deterministic functions of time and $\alpha$ is a constant.

If $\alpha = 1$ this is a linear equation, and can thus easily be solved. Now consider the case $\alpha \neq 1$ and introduce the new variable $y$ by

$$y_t = x_t^{1-\alpha}.$$

Show that $y$ satisfies the **linear** equation

$$\dot{y}_t + (1-\alpha)A_t y_t + (1-\alpha)B_t = 0.$$

**Exercise 14.3** Use the previous exercise in order to solve (14.52)–(14.53) explicitly.

**Exercise 14.4** The following example is taken from Björk *et al.* (1987). We consider a consumption problem without risky investments, but with stochastic prices for various consumption goods.

$N$ = the number of consumption goods,
$p_i(t)$ = price, at $t$, of good $i$ (measured as dollars per unit per unit time),
$p(t) = [p_1(t), \ldots, p_N(t)]'$,
$c_i(t)$ = rate of consumption of good $i$,
$c(t) = [c_1(t), \ldots, c_N(t)]'$,
$X(t)$ = wealth process,
$r$ = short rate of interest,
$T$ = time horizon.

We assume that the consumption price processes satisfy

$$dp_i = \mu_i(p)dt + \sqrt{2}\sigma_i(p)dW_i$$

where $W_1, \ldots, W_n$ are independent. The $X$-dynamics become

$$dX = rX\,dt - c'p\,dt,$$

and the objective is to maximize expected discounted utility, as measured by

$$E\left[\int_0^\tau F(t,c_t)dt\right]$$

where $\tau$ is the time of ruin, i.e.

$$\tau = \inf\{t \geq 0; X_t = 0\} \wedge T.$$

(a) Denote the optimal value function by $V(t, x, p)$ and write down the relevant HJB equation (including boundary conditions for $t = T$ and $x = 0$).
(b) Assume that $F$ is of the form

$$F(t,c) = e^{-\delta t}\prod_{i=1}^{N} c_i^{\alpha_i}$$

where $\delta > 0$, $0 < \alpha_i < 1$ and $\alpha = \sum_1^N \alpha_i < 1$. Show that the optimal value function and the optimal control have the structure

$$\begin{aligned} V(t,x,p) &= e^{-\delta t}x^{\alpha}\alpha^{-\alpha}G(t,p), \\ c_i(t,x,p) &= \frac{x}{p_i}\cdot\frac{\alpha_i}{\alpha}A(p)^{\gamma}G(t,p), \end{aligned}$$

where $G$ solves the nonlinear equation

$$\begin{cases} \dfrac{\partial G}{\partial t} + (\alpha r - \delta)G + (1-\alpha)A^{\gamma}G^{-\alpha\gamma} + \sum_i^N \mu_i \dfrac{\partial G}{\partial p_i} + \sum_i^N \sigma_i^2 \dfrac{\partial^2 G}{\partial p_i^2} & = 0, \\ G(T,p) & = 0, \; p \in R^N. \end{cases}$$

If you find this too hard, then study the simpler case when $N = 1$.

(c)Now assume that the price dynamics are given by GBM, i.e.

$$dp_i = p_i \mu_i dt + \sqrt{2} p_i \sigma_i dW_i.$$

Try to solve the $G$-equation above by making the **ansatz**

$$G(t,p) = g(t) f(p).$$

**Warning:** This becomes somwhat messy.

**Exercise 14.5** Consider as before state process dynamics

$$dX_t = \mu(t,X_t,u_t)dt + \sigma(t,X_t,u_t)dW_t$$

and the usual restrictions for $u$. Our entire derivation of the HJB equation has so far been based on the fact that the objective function is of the form

$$\int_0^T F(t,X_t,u_t)dt + \Phi(X_T).$$

Sometimes it is natural to consider other criteria, like the expected **exponential utility** criterion

$$E\left[\exp\left\{\int_0^T F(t,X_t,u_t)dt + \Phi(X_T)\right\}\right].$$

For this case we define the optimal value function as the supremum of

$$E_{t,x}\left[\exp\left\{\int_t^T F(s,X_s,u_s)dt + \Phi(X_T)\right\}\right].$$

Follow the reasoning in Section 14.3 in order to show that the HJB equation for the expected exponential utility criterion is given by

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_u \{V(t,x)F(t,x,u) + A^u V(t,x)\} & = 0, \\ V(T,x) & = e^{\Phi(x)}. \end{cases}$$

**Exercise 14.6** Solve the problem to minimize

$$E\left[\exp\left\{\int_0^T u_t^2 dt + X_T^2\right\}\right]$$

given the scalar dynamics

$$dX = (ax + u)dt + \sigma dW$$

where the control $u$ is scalar and there are no control constraints.

**Hint:** Make the ansatz

$$V(t,x) = e^{A(t)x^2 + B(t)}.$$

**Exercise 14.7** Study the general linear–exponential–qudratic control problem of minimizing

$$E\left[\exp\left\{\int_0^T \{X_t' Q X_t + u_t' R u_t\}\, dt + X_T' H X_T\right\}\right]$$

given the dynamics

$$dX_t = \{AX_t + Bu_t\}\, dt + C dW_t.$$

**Exercise 14.8** The object of this exercise is to connect optimal control to martingale theory. Consider therefore a general control problem of minimizing

$$E\left[\int_0^T F(t,X_t^{\mathbf{u}},\mathbf{u}_t)dt + \Phi(X_T^{\mathbf{u}})\right]$$

given the dynamics

$$dX_t = \mu(t,X_t,u_t)dt + \sigma(t,X_t,u_t)dW_t,$$

and the constraints

$$\mathbf{u}(t,x) \in U.$$

Now, for any control law **u** , define the **total cost process** $C(t;\mathbf{u})$ by

$$C(t, \mathbf{u}) = \int_0^t F(s,X_s^{\mathbf{u}},\mathbf{u}_s)ds + E_{t,X_t^{\mathbf{u}}}\left[\int_t^T F(s,X_s^{\mathbf{u}},\mathbf{u}_s)dt + \Phi(X_T^{\mathbf{u}})\right],$$

i.e.

$$C(t, \mathbf{u}) = \int_0^t F(s,X_s^{\mathbf{u}},\mathbf{u}_s)ds + J(t,X_t^{\mathbf{u}},\mathbf{u}).$$

Use the HJB equation in order to prove the following claims.

(a) If **u** is an arbitrary control law, then $C$ is a submartingale.
(b) If **u** is optimal, then $C$ is a martingale.

## 14.9 Notes

Standard references on otpimal control are Fleming and Rishel (1975) and Krylov (1980). A very clear exposition can be found in Øksendal (1995). For more recent work, using viscosity solutions, see Fleming and Soner (1993). The classical papers on optimal consumption are Merton (1969) and Merton (1971). See also Karatzas *et al.* (1987), and the survey paper Duffie (1994). For optimal trading under constraints, and its relation to derivative pricing see Cvitanić (1997) and references therein. See also the book by Korn (1997).

# 15 Bonds and Interest Rates

## 15.1 Zero Coupon Bonds

In this chapter we will begin to study the particular problems which appear when we try to apply arbitrage theory to the bond market. The primary objects of investigation are **zero coupon bonds** , also known as **pure discount bonds** , of various maturities. All payments are assumed to be made in a fixed currency which, for convenience, we choose to be US dollars.

**Definition 15.1** *A* **zero coupon bond** *with* **maturity date** *T, also called a T-bond, is a contract which guarantees the holder* 1 *dollar to be paid on the date T. The price at time t of a bond with maturity date T is denoted by* $p(t, T)$.

The convention that the payment at the time of maturity, known as the **principal value** or **face value** , equals one is made for computational convenience. **Coupon bonds** , which give the owner a payment stream during the interval $[0, T]$ are treated below. These instruments have the common property, that they provide the owner with a deterministic cash flow, and for this reason they are also known as **fixed income** instruments.

We now make an assumption to guarantee the existence of a sufficiently rich and regular bond market.

**Assumption 15.1.1** *We assume the following.*

- *There exists a (frictionless) market for T-bonds for every* $T>0$.
- *The relation* $p(t, t) = 1$ *holds for all t.*
- *For each fixed t, the bond price* $p(t, T)$ *is differentiable w.r.t. time of maturity T.*

Note that the relation $p(t, t) = 1$ above is necessary in order to avoid arbitrage. The bond price $p(t, T)$ is thus a stochastic object with two variables, $t$ and $T$, and, for each outcome in the underlying sample space, the dependence upon these variables is very different.

- For a fixed value of $t$, $p(t, T)$ is a function of $T$. This function provides the prices, at the fixed time $t$, for bonds of all possible maturities. The graph of this function is called "the bond price curve at $t$", or "the term structure at $t$". Typically it will be a very smooth graph, i.e. for each $t$, $p(t, T)$ will be differentiable w.r.t. $T$. The smoothness property is in fact a part of our assumptions above, but this is mainly for convenience. All models to be considered below will automatically produce smooth bond price curves.

- For a fixed maturity $T$, $p(t, T)$ (as a function of $t$) will be a scalar stochastic process. This process gives the prices, at different times, of the bond with fixed maturity $T$, and the trajectory will typically be very irregular (like a Wiener process).

We thus see that (our picture of) the bond market is different from any other market that we have considered so far, in the sense that the bond market contains an infinite number of assets (one bond type for each time of maturity). The basic goal in interest rate theory is roughly that of investigating the relations between all these different bonds. Somewhat more precisely we may pose the following general problems, to be studied below.

- What is a reasonable model for the bond market above?
- Which relations must hold between the price processes for bonds of different maturities, in order to guarantee an arbitrage free bond market?
- Is it possible to derive arbitrage free bond prices from a specification of the dynamics of the short rate of interest?
- Given a model for the bond market, how do you compute prices of interest rate derivatives, such as a European call option on an underlying bond?

## 15.2 Interest Rates

### 15.2.1 Definitions

Given the bond market above, we may now define a number of interest rates, and the basic construction is as follows. Suppose that we are standing at time $t$, and let us fix two other points in time, $S$ and $T$, with $t < S < T$. The immediate project is to write a contract at time $t$ which allows us to make an investment of one (dollar) at time $S$, and to have a **deterministic** rate of return, determined at the contract time $t$, over the interval $[S, T]$. This can easily be achieved as follows.

1. At time $t$ we sell one $S$-bond. This will give us $p(t, S)$ dollars.
2. We use this income to buy exactly $p(t, S)/p(t, T)$ $T$-bonds. Thus our net investment at time $t$ equals zero.
3. At time $S$ the $S$-bond matures, so we are obliged to pay out one dollar.
4. At time $T$ the $T$-bonds mature at one dollar a piece, so we will receive the amount $p(t, S)/p(t, T)$ dollars.
5. The net effect of all this is that, based on a contract at $t$, an investment of one dollar at time $S$ has yielded $p(t, S)/p(t, T)$ dollars at time $T$.
6. Thus, at time $t$, we have made a contract guaranteeing a **riskless** rate of interest over the **future interval** $[S, T]$. Such an interest rate is called a **forward rate** .

We now go on to compute the relevant interest rates implied by the construction above. We will use two (out of many possible) ways of quoting forward rates, namely as continuously compounded rates or as simple rates.

The **simple** forward rate (or **LIBOR** rate) $L$, is the solution to the equation

$$1 + (T - S)L = \frac{p(t,S)}{p(t,T)},$$

whereas the **continuously compounded** forward rate $R$ is the solution to the equation

$$e^{R(T-S)} = \frac{p(t,S)}{p(t,T)}.$$

The simple rate notation is the one used in the market, whereas the continuously compounded notation is used in theoretical contexts. They are of course logically equivalent, and the formal definitions are as follows.

**Definition 15.2**

1. *The simple* **forward rate for [*S*, *T*] contracted at** *t, henceforth referred to as the* **LIBOR** *forward rate, is defined as*

$$L(t, S,T) = -\frac{p(t,T) - p(t,S)}{(T - S)p(t,T)}.$$

2. *The simple* **spot rate for [*S*, *T*]** *, henceforth referred to as the* **LIBOR** *spot rate, is defined as*

$$L(S,T) = -\frac{p(S,T) - 1}{(T - S)p(S,T)}.$$

3. *The continuously compounded* **forward rate for [*S*, *T*] contracted at** *t is defined as*

$$R(t, S,T) = -\frac{\log p(t,T) - \log p(t,S)}{T - S}.$$

4. *The* **continuously compounded spot rate** , $R(S, T)$, *for the period* $[S, T]$ *is defined as*

$$R(S,T) = \frac{\log p(S,T)}{T - S}.$$

5. *The* **instantaneous forward rate with maturity** $T$, **contracted at** *t, is defined by*

$$f(t,T) = \frac{\partial \log p(t,T)}{\partial T}.$$

6. *The instantaneous* **short rate at time** *t is defined by*

$$r(t) = f(t,t).$$

We note that spot rates are forward rates where the time of contracting coincides with the start of the interval over which the interest rate is effective, i.e. $t = S$. The instantaneous forward rate, which will be of great importance below, is the limit of the continuously compounded forward rate when $S \to T$. It can thus be interpreted as the riskless rate of interest, contracted at $t$, over the infinitesimal interval $[T, T + dT]$.

We now go on to define the money account process $B$.

**Definition 15.3** *The* **money account** *process is defined by*

$$B_t = \exp\left\{\int_0^t r(s)ds\right\},$$

*i.e.*

$$\begin{cases} dB(t) &= r(t)B(t)dt, \\ B(0) &= 1. \end{cases}$$

The interpretation of the money account is the same as before, i.e. you may think of it as describing a bank with a stochastic short rate of interest. It can also be shown (see below) that investing in the money account is equivalent to investing in a self-financing "rolling over" trading strategy, which at each time $t$ consists entirely of "just maturing" bonds, i.e. bonds which will mature at $t + dt$.

As an immediate consequence of the definitions we have the following useful formulas.

**Lemma 15.4** *For $t \leq s \leq T$ we have*

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

*and in particular*

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

If we wish to make a model for the bond market, it is obvious that this can be done in many different ways.

- We may specify the dynamics of the short rate (and then perhaps try to derive bond prices using arbitrage arguments).
- We may directly specify the dynamics of all possible bonds.
- We may specify the dynamics of all possible forward rates, and then use Lemma 15.4 in order to obtain bond prices.

All these approaches are of course related to each other, and we now go on to present a small "toolbox" of results to facilitate the analysis below. These results will not be used until Chapter 18, and the proofs are somewhat technical, so the next two subsections can be omitted at a first reading.

### 15.2.2 Relations Between df(t, T), dp(t, T) and dr(t)

We will consider dynamics of the following form.

**Short rate dynamics**

$$dr(t) = a(t)dt + b(t)dW(t).$$

**Bond price dynamics** (15.1)

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t).$$

**Forward rate dynamics** (15.2)

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t).$$

(15.3)

The Wiener process $W$ is allowed to be vector valued, in which case the volatilities $v(t, T)$ and $\sigma(t, T)$ are row vectors. The processes $a(t)$ and $b(t)$ are scalar adapted processes, whereas $m(t, T)$, $v(t, T)$, $\alpha(t, T)$ and $\sigma(t, T)$ are adapted processes parameterized by time of maturity $T$. The interpretation of the bond price equation (15.2) and the forward rate equation (15.3) is that these are scalar stochastic differential equations (in the $t$-variable) for each fixed time of maturity $T$. Thus (15.2) and (15.3) are both infinite dimensional systems of SDEs.

We will study the formal relations which must hold between bond prices and interest rates, and to this end we need a number of technical assumptions, which we collect below in an "operational" manner.

**Assumption 15.2.1**

1. *For each fixed* $\omega$, $t$ *all the objects* $m(t, T)$, $v(t, T)$, $\alpha(t, T)$ *and* $\sigma(t, T)$ *are assumed to be continuously differentiable in the T-variable. This partial T-derivative is sometimes denoted by* $m_T(t, T)$ *etc.*
2. *All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.*

The main result is as follows. Note that the results below hold, regardless of the measure under consideration, and in particular we do **not** assume that markets are free of arbitrage.

**Proposition 15.5**

1. *If* $p(t, T)$ *satisfies (15.2), then for the forward rate dynamics we have*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t).$$

where $\alpha$ and $\sigma$ are given by

$$\begin{cases} \alpha(t,T) &= vT(t,T) \cdot v(t,T) - m_T(t,T), \\ \sigma(t,T) &= -v_T(t,T). \end{cases}$$

2. *If* $f(t, T)$ *satisfies (15.3) then the short rate satisfies* (15.4)

$$dr(t) = a(t)dt + b(t)dW(t),$$

*where*

$$\begin{cases} a(t) &= f_T(t,t) + \alpha(t,t), \\ b(t) &= \sigma(t,t). \end{cases}$$

(15.5)

3. *If $f(t, T)$ satisfies (15.3) then $p(t, T)$ satisfies*

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\left\|S(t,T)\right\|^2\right\}dt + p(t,T)S(t,T)dW(t),$$

*where $\|\cdot\|$ denotes the Euclidean norm, and*

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds. \end{cases}$$

(15.6)

**Proof** The first part of the proposition is left to the reader (see the exercises). For the second part we integrate the forward rate dynamics to get

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s).$$

Now we can write

(15.7)

$$\begin{aligned} \alpha(s,t) &= \alpha(s,s) + \int_s^t \alpha_T(s,u)du, \\ \sigma(s,t) &= \sigma(s,s) + \int_s^t \sigma_T(s,u)du, \end{aligned}$$

and, inserting this into (15.7), we have

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,s)ds + \int_0^t\int_s^t \alpha_T(s,u)duds \\ &+ \int_0^t \sigma(s,s)dW_s + \int_0^t\int_s^t \sigma_T(s,u)dudW_s. \end{aligned}$$

Changing the order of integration and identifying terms we obtain the result.

For the proof of the third part we give a slightly heuristic argument. The full formal proof, see Heath *et al.* (1987), is an integrated version of the proof given here, but the infinitesimal version below is (hopefully) easier to understand. Using the definition of the forward rates we may write

$$p(t,T) = e^{Y(t,T)},$$

where $Y$ is given by

(15.8)

$$Y(t,T) = -\int_t^T f(t,s)ds.$$

From the Itô formula we then obtain the bond dynamics as

(15.9)

$$dp(t,T) = p(t,T)dY(t,T) + \frac{1}{2}p(t,T)\left(dY(t,T)\right)^2,$$

(15.10)

and it remains to compute $dY(t, T)$. We have

$$dY(t,T) = -d\left(\int_t^T f(t,s)ds\right),$$

and the problem is that in the integral the $t$-variable occurs in two places: as the lower limit of integration, and in the integrand $f(t, s)$. This is a situation that is not covered by the standard Itô formula, but it is easy to guess the answer. The $t$ appearing as the lower limit of integration should give rise to the term

$$\frac{\partial}{\partial t}\left(\int_t^T f(t,s)ds\right)dt.$$

Furthermore, since the stochastic differential is a linear operation, we should be allowed to move it inside the integral, thus providing us with the term

$$\left(\int_t^T df(t,s)ds\right).$$

We have therefore arrived at

$$dY(t,T) = -\frac{\partial}{\partial t}\left(\int_t^T f(t,s)ds\right)dt - \int_t^T df(t,s)ds,$$

which, using the fundamental theorem of integral calculus, as well as the forward rate dynamics, gives us

$$dY(t,T) = f(t,t)dt - \int_t^T \alpha(t,s)dtds - \int_t^T \sigma(t,s)dW_t ds.$$

We now exchange $dt$ and $dW_t$ with $ds$ and recognize $f(t, t)$ as the short rate $r(t)$, thus obtaining

$$dY(t,T) = r(t)dt + A(t,T)dt + S(t,T)dW_t,$$

with $A$ and $S$ as above. We therefore have

$$(dY(t,T))^2 = \left\|S(t,T)\right\|^2 dt,$$

and, substituting all this into (15.10), we obtain our desired result.

## 15.2.3 An Alternative View of the Money Account

The object of this subsection is to show (heuristically) that the risk free asset $B$ can in fact be replicated by a self-financing strategy, defined by “rolling over” just-maturing bonds. This is a “folklore” result, which is very easy to prove in discrete time, but surprisingly tricky in a continuous time framework.

Let us consider a self-financing portfolio which at each time $t$ consists entirely of bonds maturing $x$ units of time later (where we think of $x$ as a small number). At time $t$ the portfolio thus consists only of bonds with maturity $t + x$, so the value dynamics for this portfolio is given by

$$dV(t) = V(t) \cdot 1 \cdot \frac{dp(t,t+x)}{p(t,t+x)}, \tag{15.11}$$

where the constant 1 indicates that the weight of the $t + x$-bond in the portfolio equals one. We now want to study the behavior of this equation as $x$ tends to zero, and to this end we use Proposition 15.5 to obtain

$$\frac{dp(t,t+x)}{p(t,t+x)} = \left\{r(t) + A(t,t+x) + \frac{1}{2}\left\|S(t,t+x)\right\|^2\right\}dt + S(t,t+x)dW(t).$$

Letting $x$ tend to zero, (15.6) gives us

$$\lim_{x \to 0} A(t,t+x) = 0,$$
$$\lim_{x \to 0} S(t,t+x) = 0.$$

Furthermore we have

$$\lim_{x \to 0} p(t,t+x) = 1,$$

and, substituting all this into eqn (15.11), we obtain the value dynamics

$$dV(t) = r(t)V(t)dt, \tag{15.12}$$

which we recognize as the dynamics of the money account.

The argument thus presented is of course only heuristical, and it requires some hard work to make it precise. Note, for example, that the rolling over portfolio above does not fall into the general framework of self-financing portfolios, developed earlier. The problem is that, although at each time $t$, the portfolio only consists of one particular bond (maturing at $t + x$), over an arbitrary short time interval, the portfolio will use an infinite number of different bonds. In order to handle such a situation, we need to extend the portfolio concept to include measure valued portfolios. This is done in Björk *et al.* (1997a), and in Björk *et al.* (1997b) the argument above is made precise.

## 15.3 Coupon Bonds, Swaps and Yields

In most bond markets, there are only a relative small number of zero coupon bonds traded actively. The maturities for these are generally short (typically between half a year and two years), whereas most bonds with a longer time to maturity are coupon bearing. Despite this empirical fact we will still assume the existence of a market for all possible pure discount bonds, and we now go on to introduce and price coupon bonds in terms of zero coupon bonds.

## 15.3.1 Fixed Coupon Bonds

The simplest coupon bond is the **fixed coupon bond** . This is a bond which, at some intermediary points in time, will provide predetermined payments (coupons) to the holder of the bond. The formal description is as follows.

- Fix a number of dates, i.e. points in time, $T_0, \dots, T_n$. Here $T_0$ is interpreted as the emission date of the bond, whereas $T_1, \dots, T_n$ are the coupon dates.
- At time $T_i$, $i = 1, \dots, n$, the owner of the bond receives the deterministic coupon $c_i$.
- At time $T_n$ the owner receives the face value $K$.

We now go on to compute the price of this bond, and it is obvious that the coupon bond can be replicated by holding a portfolio of zero coupon bonds with maturities $T_i$, $i = 1, \dots, n$. More precisely we will hold $c_i$ zero coupon bonds of maturity $T_i$, $i = 1, \dots, n - 1$, and $K + c_n$ bonds with maturity $T_n$, so the price, $p(t)$, at a time $t < T_1$, of the coupon bond is given by

$$p(t) = K \cdot p(t,T_n) + \sum_{i=1}^{n} c_i \cdot p(t,T_i). \tag{15.13}$$

Very often the coupons are determined in terms of **return** , rather than in monetary (e.g. dollar) terms. The return for the $i$th coupon is typically quoted as a simple rate acting on the face value $K$, over the period $[T_{i-1}, T_i]$. Thus, if, for example, the $i$th coupon has a return equal to $r_i$, and the face value is $K$, this means that

$$c_i = r_i(T_i - T_{i-1})K.$$

For a standardized coupon bond, the time intervals will be equally spaced, i.e.

$$T_i = T_0 + i\delta,$$

and the coupon rates $r_1, \dots, r_n$ will be equal to a common coupon rate $r$. The price $p(t)$ of such a bond will, for $t \leq T_1$, be given by

$$p(t) = K \cdot \left( p(t,T_n) + r\delta \sum_{i=1}^{n} \cdot p(t,T_i) \right). \tag{15.14}$$

## 15.3.2 Floating Rate Bonds

There are various coupon bonds for which the value of the coupon is not fixed at the time the bond is issued, but rather reset for every coupon period. Most often the resetting is determined by some financial benchmark, like a market interest rate, but there are also bonds for which the coupon is benchmarked against a nonfinancial index.

As an example (to be used in the context of swaps below), we will confine ourselves to discussing one of the simplest floating rate bonds, where the coupon rate $r_i$ is set to the spot LIBOR rate $L(T_{i-1}, T_i)$. Thus

$$c_i = (T_i - T_{i-1})L(T_{i-1},T_i)K,$$

and we note that $L(T_{i-1}, T_i)$ is determined already at time $T_{i-1}$, but that $c_i$ is not delivered until at time $T_i$. We now go on to compute the value of this bond at some time $t < T_0$, in the case when the coupon dates are equally spaced, with $T_i - T_{i-1} = \delta$, and to this end we study the individual coupon $c_i$. Without loss of generality we may assume that $K = 1$, and inserting the definition of the LIBOR rate (Definition 15.2) we have

$$c_i = \delta \frac{1 - p(T_{i-1},T_i)}{\delta p(T_{i-1},T_i)} = \frac{1}{p(T_{i-1},T_i)} - 1.$$

The value at $t$, of the term $-1$ (paid out at $T_i$), is of course equal to

$$-p(t,T_i),$$

and it remains to compute the value of the term $\frac{1}{p(T_{i-1},T_i)}$, which is paid out at $T_i$.

This is, however, easily done through the following argument.

- Buy, at time $t$, one $T_{i-1}$-bond. This will cost $p(t, T_{i-1})$.
- At time $T_{i-1}$ you will receive the amount 1.
- Invest this unit amount in $T_i$-bonds. This will give you exactly $\frac{1}{p(T_{i-1},T_i)}$ bonds.
- At $T_i$ the bonds will mature, each at the face value 1. Thus, at time $T_i$, you will obtain the amount

$$\frac{1}{p(T_{i-1},T_i)}.$$

This argument shows that it is possible to replicate the cash flow above, using a self-financing bond strategy, to the initial cost $p(t, T_{i-1})$. Thus the value at $t$, of obtaining $\frac{1}{p(T_{i-1},T_i)}$, at $T_i$, is given by $p(t, T_{i-1})$, and the value at $t$ of the coupon $c_i$ is

$$p(t,T_{i-1}) - p(t,T_i).$$

Summing up all the terms we finally obtain the following valuation formula for the floating rate bond

$$p(t) = p(t,T_n) + \sum_{i=1}^{n} [p(t,T_{i-1}) - p(t,T_i)] = p(t,T_0). \tag{15.15}$$

In particular we see that if $t = T_0$, then $p(T_0) = 1$. The reason for this (perhaps surprisingly easy) formula is of course that the entire floating rate bond can be replicated through a self-financing portfolio (see the exercises).

## 15.3.3 Interest Rate Swaps

In this section we will discuss the simplest of all interest rate derivatives, the interest rate swap. This is basically a scheme where you exchange a payment stream at a fixed rate of interest, known as the **swap rate** , for a payment stream at a floating rate (typically a LIBOR rate).

There are many versions of interest rate swaps, and we will study the **forward swap settled in arrears** , which is defined as follows. We denote the principal by $K$, and the swap rate by $R$. By assumption we have a number of equally spaced dates $T_0, \dots, T_n$, and payment occurs at the dates $T_1, \dots, T_n$ (not at $T_0$). If you swap a fixed rate for a floating rate (in this case the LIBOR spot rate), then, at time $T_i$, you will receive the amount

$$K\delta L(T_{i-1},T_i),$$

which is exactly $Kc_i$, where $c_i$ is the $i$th coupon for the floating rate bond in the previous section. At $T_i$ you will pay the amount

$$K\delta R.$$

The net cash flow at $T_i$ is thus given by

$$K\delta[L(T_{i-1},T_i)-R],$$

and using our results from the floating rate bond, we can compute the value at $t < T_0$ of this cash flow as

$$Kp(t,T_{i-1})-K(1+\delta R)p(t,T_i).$$

The total value $\Pi(t)$, at $t$, of the swap is thus given by

$$\Pi(t)=K\sum_{i=1}^{n}[p(t,T_{i-1})-(1+\delta R)p(t,T_i)],$$

and we can simplify this to obtain the following result.

**Proposition 15.6** *The price, for $T_0 < t < T_1$, of the swap above is given by*

$$\Pi(t)=Kp(t,T_0)-K\sum_{i=1}^{n}d_i p(t,T_i),$$

*where*

$$\begin{aligned} d_i &= R\delta,\ i=1,\dots,n-1,\\ d_n &= 1+R\delta. \end{aligned}$$

The remaining question is how the swap rate $R$ is determined. By definition it is chosen such that the value of the swap equals zero at the time when the contract is made. We have the following easy result.

**Proposition 15.7** *If, by convention, we assume that the contract is written at $t = 0$, the swap rate is gven by*

$$R = \frac{p(0,T_0) - p(0,T_n)}{\delta\sum_1^n p(0,T_i)}.$$

*In the case that $T_0 = 0$ this formula reduces to*

$$R = \frac{1 - p(0,T_n)}{\delta\sum_1^n p(0,T_i)}.$$

## 15.3.4 Yield and Duration

Consider a zero coupon $T$-bond with market price $p(t, T)$. We now look for the bond's "internal rate of interest", i.e. the constant short rate of interest which will give the same value to this bond as the value given by the market. Denoting this value of the short rate by $y$, we thus want to solve the equation

$$p(t,T) = e^{-y\cdot(T-t)}\cdot 1,$$

where the factor 1 indicates the face value of the bond. We are thus led to the following definition.

**Definition 15.8** *The continuously compounded* **zero coupon yield** , $y(t, T)$, *is given by*

$$y(t,T) = -\frac{\log p(t,T)}{T-t}.$$

*For a fixed t, the function $T \mapsto y(t, T)$ is called the (zero coupon)* **yield curve** .

We note that the yield $y(t, T)$ is nothing more than the spot rate for the interval $[t, T]$. Now let us consider a fixed coupon bond of the form discussed in Section 15.3.1 where, for simplicity of notation, we include the face value in the coupon $c_n$. We denote its market value at $t$ by $p(t)$. In the same spirit as above we now look for its internal rate of interest, i.e. the constant value of the short rate, which will give the market value of the coupon bond.

**Definition 15.9** *The* **yield to maturity** , $y(t, T)$, *of a fixed coupon bond at time t, with market price p, and payments $c_i$ at $T$ for $i = 1, \ldots, n$, is defined as the value of y which solves the equation*

$$p(t) = \sum_{i=1}^{n} c_i e^{-y(T_i-t)}.$$

An important concept in bond portfolio management is the "Macaulay duration". Without loss of generality we may assume that $t = 0$.

**Definition 15.10** *For the fixed coupon bond above, with price p at $t = 0$, and yield to maturity y, the* **duration** , *D, is defined as*

$$D = \frac{\sum_1^n T_i c_i e^{-yT_i}}{p}.$$

The duration is thus a weighted average of the coupon dates of the bond, where the discounted values of the coupon payments are used as weights, and it will in a sense provide you with the "mean time to coupon payment". As such it is an important concept, and it also acts a measure of the sensitivity of the bond price w.r.t. changes in the yield. This is shown by the following obvious result.

**Proposition 15.11** *With notation as above we have*

$$\frac{dp}{dy} = \frac{d}{dy}\left\{\sum_{1}^{n} c_i e^{-yT_i}\right\} = -D \cdot p. \tag{15.16}$$

Thus we see that duration is essentially for bonds (w.r.t. yield) what delta (see Section 8.2) is for derivatives (w.r.t. the underlying price). The bond equivalent of the gamma is **convexity** , which is defined as

$$C = \frac{\partial^2 p}{\partial y^2}.$$

## 15.4 Exercises

**Exercise 15.1 A forward rate agreement** (FRA) is a contract, by convention entered into at $t = 0$, where the parties (a lender and a borrower) agree to let a certain interest rate, $R^*$, act on a prespecified principal, $K$, over some future period $[S, T]$. Assuming that the interest rate is continuously compounded, the cash flow to the lender is, by definition, given as follows:

- At time $S$: $-K$.
- At time $T$: $Ke^{R^*(T-S)}$.

The cash flow to the borrower is of course the negative of that to the lender.

(a) Compute for any time $t < S$, the value, $\Pi(t)$, of the cash flow above in terms of zero coupon bond prices.
(b) Show that in order for the value of the FRA to equal zero at $t = 0$, the rate $R^*$ has to equal the forward rate $R(0; S, T)$ (compare this result to the discussion leading to the definition of forward rates).

**Exercise 15.2** Prove the first part of Proposition 15.5.

**Hint:** Apply the Itô formula to the process $\log p(t, T)$, write this in integrated form and differentiate with respect to $T$.

**Exercise 15.3** Consider a coupon bond, starting at $T_0$, with face value $K$, coupon payments at $T_1, \dots, T_n$ and a fixed coupon rate $r$. Determine the coupon rate $r$, such that the price of the bond, at $T_0$, equals its face value.

**Exercise 15.4** Derive the pricing formula (15.15) directly, by constructing a self-financing portfolio which replicates the cash flow of the floating rate bond.

**Exercise 15.5** Let $\{y(0, T); T \geq 0\}$ denote the zero coupon yield curve at $t = 0$. Assume that, apart from the zero coupon bonds, we also have exactly one fixed coupon bond for every maturity $T$. We make no particular assumptions about the coupon bonds, apart from the fact that all coupons are positive, and we denote the yield to maturity, again at time $t = 0$, for the coupon bond with maturity $T$, by $y_M(0, T)$. We now have three curves to consider: the forward rate curve $f(0, T)$, the zero coupon yield curve $y(0, T)$, and the coupon yield curve $y_M(0, T)$. The object of this exercise is to see how these curves are connected.

(a) Show that

$$f(0,T) = y(0,T) + T \cdot \frac{\partial y(0,T)}{\partial T}.$$

(b) Assume that the zero coupon yield cuve is an increasing function of $T$. Show that this implies the inequalities

$$y_M(0,T) \leq y(0,T) \leq f(0,T), \ \forall T,$$

(with the opposite inequalities holding if the zero coupon yield curve is decreasing). Give a verbal economic explanation of the inequalities.

**Exercise 15.6** Prove Proposition 15.11.

**Exercise 15.7** Consider a **consol bond** , i.e. a bond which will forever pay one unit of cash at $t = 1, 2, \ldots$. Suppose that the market yield $y$ is constant for all maturities.

(a) Compute the price, at $t = 0$, of the consol.
(b) Derive a formula (in terms of an infinite series) for the duration of the consol.
(c) Use (a) and Proposition 15.11 in order to compute an analytical formula for the duration.
(d) Compute the convexity of the consol.

# 15.5 Notes

Fabozzi (1995), and Sundaresan (1997) are standard textbooks on bond markets.

# 16 Short Rate Models

## 16.1 Generalities

In this chapter we turn to the problem of how to model an arbitrage free family of zero coupon bond price processes $\{p(\cdot,T); T \geq 0\}$ .

Since, at least intuitively, the price, $p(t,T)$, should in some sense depend upon the behavior of the short rate of interest over the interval $[t,T]$, a natural starting point is to give an a priori specification of the dynamics of the short rate of interest. This has in fact been the "classical" approach to interest rate theory, so let us model the short rate, under the objective probability measure $P$, as the solution of an SDE of the form

$$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))d\overline{W}(t). \tag{16.1}$$

The short rate of interest is the only object given a priori, so the only exogenously given asset is the money account, with price process $B$ defined by the dynamics

$$dB(t) = r(t)B(t)dt. \tag{16.2}$$

As usual we interpret this as a model of a bank with the stochastic short rate of interest $r$. The dynamics of $B$ can then be interpreted as the dynamics of the value of a bank account. To be quite clear let us formulate the above as a formalized assumption.

**Assumption 16.1.1** *We assume the existence of one exogenously given (locally risk free) asset. The price, B, of this asset has dynamics given by eqn (16.2), where the dynamics of r, under the objective probability measure P, are given by eqn (16.1).* As in the previous chapter, we make an assumption to guarantee the existence of a sufficiently rich bond market.

**Assumption 16.1.2** *We assume that there exists a market for zero coupon T-bonds for every value of T.* We thus assume that our market contains all possible bonds (plus, of course, the risk free asset above). Consequently it is a market containing an infinite number of assets, but we again stress the fact that only the risk free asset is exogenously given. In other words, in this model the risk free asset is considered as the underlying asset whereas all bonds are regarded as derivatives of the "underlying" short rate $r$. Our main goal is broadly to investigate the relationship which must hold in an arbitrage free market between the price processes of bonds with different maturities. As a second step we also want to obtain arbitrage free prices for other interest rate derivatives such as bond options and interest rate swaps.

Since we view bonds as interest rate derivatives it is natural to ask whether the bond prices are uniquely determined by the given $r$ dynamics in (16.1) and the Condition that the bond market shall be free of arbitrage. This question, and its answer, are fundamental.

**Question:**

**Are bond prices uniquely determined by the $P$-dynamics of the short rate $r$?**

**Answer:**

**No!**

For the reader who has studied Chapter 10, this negative result should be fairly obvious. The arguments below are parallel to those of Section 10.2, and the results are in fact special cases of the general results in Section 10.4. If you have already studied these sections you can thus browse quickly through the text until the term structure equation (16.2). In order to keep this part of the book self-contained, and since the discussion is so important, we will (with some apologies) give the full argument.

Let us start by viewing the bond market in the light of the meta-theorem 7.3.1. We see that in the present situation the number $M$ of exogenously given traded assets **excluding** the risk free asset equals zero. The number $R$ of random sources on the other hand equals one (we have one driving Wiener process). From the meta-theorem we may thus expect that the exogenously given market is arbitrage free but not complete. The lack of completeness is quite clear: since the only exogenously given asset is the risk free one we have no possibility of forming interesting portfolios. The only thing we can do on the a priori given market is simply to invest our initial capital in the bank and then sit down and wait while the portfolio value evolves according to the dynamics (16.2). It is thus impossible to replicate an interesting derivative, even such a simple one as a $T$-bond.

Another way of seeing this problem appears if we try to price a certain $T$-bond using the technique used in Section 6.3. In order to imitate the old argument we would assume that the price of a certain bond is of the form $F(t,r(t))$.

Then we would like to form a risk free portfolio based on this bond and on the underlying asset. The rate of return of this risk free portfolio would then, by an arbitrage argument, have to equal the short rate of interest, thus giving us some kind of equation for the determination of the function $F$. Now, in the Black-Scholes model the underlying asset was the stock $S$, and at first glance this would correspond to $r$ in the present situation. Here, however, we have the major difference between the Black-Scholes model and our present model. The short rate of interest $r$ is **not** the price of a **traded** asset, i.e. there is no asset on the market whose price process is given by $r$. Thus it is meaningless to form a portfolio "based on $r$". Since there sometimes is a lot of confusion on this point let us elaborate somewhat. We observe then that the English word "price" can be used in two related but different ways.

The first way occurs in everyday (informal) speech, and in this context it is not unusual (or unreasonable) to say that the short rate of interest reflects the price of borrowing money in the bank. In particular we often say that it is expensive to borrow money if the rate of interest is high, and cheap when the rate of interest is low.

The second (formalized) use of the word "price" occurs when we are dealing with price systems in the context of, for example, general equilibrium theory. In this setting the word "price" has a much more precise and technical meaning than in everyday language. Firstly a price is now measured in a **unit** like, say, pounds sterling. The short rate of interest, on the contrary, is measured in the unit $(time)^{-1}$, though for numerical reasons it is sometimes given as a precentage. Secondly the price of an asset tells you how many pounds sterling you have to pay for one unit of the asset in question. If, say, the price of *ACME INC.* stock is 230 pounds this means that if you pay 230 pounds then you will obtain one share in *ACME INC.* If, on the other hand, the short rate of interest is 11%, this does **not** mean that you can pay 11 (units of what?) in order to obtain one unit of some asset (what would that be?).

This does not at all imply that the everyday interpretation of the interest rate as "the price of borrowing money" is wrong. This aspect of the short rate already appears in fact in the equation, $dB = rBdt$, for the money account, where it is obvious that if $r$ is high, then our debt to the bank grows at a high rate.

When we use the word "price" in this text it is exclusively as in the second formalized meaning above, and a sloppy usage will easily lead to nonsense and chaos.

To sum up:

- The price of a particular bond will **not** be completely determined by the specification (16.1) of the $r$-dynamics and the requirement that the bond market is free of arbitrage.
- The reason for this fact is that arbitrage pricing is always a case of pricing a derivative **in terms of** the price of some underlying assets. In our market we do not have sufficiently many underlying assets.

We thus fail to determine a unique price of a particular bond. Fortunately this (perhaps disappointing) fact does not mean that bond prices can take any form whatsoever. On the contrary we have the following basic intuition.

### Idea 16.1.1

- *Prices of bonds with* **different maturities** *will have to satisfy certain* **internal consistency relations** *in order to avoid arbitrage possibilities on the bond market.*
- *If we take the price of* **one** *particular "benchmark" bond as given then the prices of* **all other** *bonds will be uniquely determined* **in terms of** *the price of the benchmark bond (and the r-dynamics).*

This fact is in complete agreement with the meta-theorem, since in the a priori given market consisting of one benchmark bond plus the risk free asset we will have $R = M = 1$ thus guaranteeing completeness.

# 16.2 The Term Structure Equation

To make the ideas presented in the previous section more concrete we now begin our formal treatment.

**Assumption 16.2.1** *We assume that there is a market for T-bonds for every choice of T and that the market is arbitrage free. We assume furthermore that, for every T, the price of a T-bond has the form*

$$p(t,T) = F(t,r(t);T), \tag{16.3}$$

*where F is a smooth function of three real variables.*

Conceptually it is perhaps easiest to think of $F$ as a function of only two variables, namely $r$ and $t$, whereas $T$ is regarded as a parameter. Sometimes we will therefore write $F^T(t,r)$ instead of $F(t,r;T)$. The main problem now is to find out what $F^T$ may look like on an arbitrage free market.

Just as in the case of stock derivatives we have a simple boundary Condition. At the time of maturity a $T$-bond is of course worth exactly 1 pound, so we have the relation

$$F(T,r,T) = 1, \quad \text{for all } r. \tag{16.4}$$

Note that in the equation above the letter $r$ denotes a real variable, while at the same time $r$ is used as the name of the stochastic process for the short rate. To conform with our general notational principles we should really denote the stochastic process by a capital letter like $R$, and then denote an outcome of $R$ by the letter $r$. Unfortunately the use of $r$ as the name of the stochastic process seems to be so fixed that it cannot be changed. We will thus continue to use $r$ as a name both for the process and for a generic outcome of the process. This is somewhat sloppy, but we hope that the meaning will be clear from the context.

In order to implement the ideas above we will now form a portfolio consisting of bonds having different times of maturity. We thus fix two times of maturity $S$

and $T$. From Assumption 16.2.1 and the Itô formula we get the following price dynamics for the $T$-bond, with corresponding equations for the $S$-bond.

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\overline{W},$$

where, with subindices $r$ and $t$ denoting partial derivatives, (16.5)

$$\alpha_T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T},$$

$$\sigma_T = \frac{\sigma F_r^T}{F^T}. \tag{16.6}$$

Denoting the relative portfolio by $(u_S, u_T)$ we have the following value dynamics for our portfolio. (16.7)

$$dV = V\left\{u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S}\right\},$$

(16.8)

and inserting the differential from (16.5), as well as the corresponding equation for the $S$-bond, gives us, after some reshuffling of terms,

$$dV = V \cdot \{u_T\alpha_T + u_S\alpha_S\}\, dt + V \cdot \{u_T\sigma_T + u_S\sigma_S\}\, d\overline{W}.$$

Exactly as in Section 6.3 we now define our portfolio by the equations (16.9)

$$u_T + u_S = 1,$$

$$u_T\sigma_T + u_S\sigma_S = 0. \tag{16.10}$$

With this portfolio the $d\overline{W}$-term in (16.9) will vanish, so the value dynamics reduce to (16.11)

$$dV = V \cdot \{u_T\alpha_T + u_S\alpha_S\}\, dt.$$

The system (16.10)–(16.11) can easily be solved as (16.12)

$$u_T = -\frac{\sigma_S}{\sigma_T - \sigma_S},$$

$$u_S = \frac{\sigma_T}{\sigma_T - \sigma_S}, \tag{16.13}$$

and substituting this into (16.12) gives us (16.14)

$$dV = V \cdot \left\{\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right\} dt.$$

(16.15)

Using Proposition 6.6, the assumption of no arbitrage now implies that this portfolio must have a rate of return equal to the short rate of interest. Thus we have the Condition

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r(t), \quad \text{for all} \quad t, \text{with probability} \quad 1,$$

(16.16)

or, written differently,

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}. \tag{16.17}$$

The interesting fact about eqn (16.17) is that on the left hand side we have a stochastic process which does not depend on the choice of $T$, whereas on the right hand side we have a process which does not depend on the choice of $S$. The common quotient will thus not depend on the choice of either $T$ or $S$, so we have thus proved the following fundamental result.

**Proposition 16.1** *Assume that the bond market is free of arbitrage. Then there exists a process* $\lambda$ *such that the relation*

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t) \tag{16.18}$$

*holds for all* $t$ *and for every choice of maturity time* $T$.

Observe that the process $\lambda$ is **universal** in the sense that it is the same $\lambda$ which occurs on the right hand side of (16.18) regardless of the choice of $T$. Let us now take a somewhat closer look at this process.

In the numerator of (16.18) we have the term $\alpha_T(t) - r(t)$. By eqn (16.5), $\alpha_T(t)$ is the local rate of return on the $T$-bond, whereas $r$ is the rate of return of the risk free asset. The difference $\alpha_T(t) - r(t)$ is thus the **risk premium** of the $T$-bond. It measures the excess rate of return for the risky $T$-bond over the riskless rate of return which is required by the market in order to avoid arbitrage possibilities. In the denominator of (16.18) we have $\sigma_T(t)$, i.e. the local volatility of the $T$-bond.

Thus we see that the process $\lambda$ has the dimension "risk premium per unit of volatility". The process $\lambda$ is known as the **market price of risk** , and we can paraphrase Proposition 16.1 by the following slogan.

- **In a no arbitrage market all bonds will, regardless of maturity time, have the same market price of risk.**

Before we move on, a brief word of warning: the name "market price of risk" is in some sense rather appealing and reasonable, but it is important to realize that the market price of risk is **not** a price in the technical (general equilibrium) sense reserved for the word "price" in the rest of this text. We do not measure $\lambda$ in SEK, and $\lambda$ is not something which we pay in order to obtain some commodity. Thus the usage of the word "price" in this context is that of informal everyday language, and one should be careful not to overinterpret the **words** "market price of risk" by assuming that properties holding for price processes in general equilibrium theory also automatically hold for the process $\lambda$.

We may obtain even more information from eqn (16.18) by inserting our earlier formulas (16.6)–(16.7) for $\alpha_T$ and $\sigma_T$. After some manipulation we then obtain one of the most important equations in the theory of interest rates—the so called "term structure equation". Since this equation is so fundamental we formulate it as a separate result.

**Proposition 16.2 (Term structure equation)** *In an arbitrage free bond market, $F^T$ will satisfy the term structure equation*

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\ F^T(T,r) &= 1. \end{cases}$$

(16.19)

The term structure equation is obviously closely related to the Black-Scholes equation, but it is a more complicated object due to the appearance of the market price of risk λ. It follows from eqns (16.6), (16.7) and (16.18) that λ is of the form $\lambda = \lambda(t,r)$ so the term structure equation is a standard PDE, but the problem is that λ is not determined within the model. In order to be able to solve the term structure equation we must specify λ exogenously just as we have to specify μ and σ.

Despite this problem it is not hard to obtain a Feynman-Kač representation of $F^T$. This is done by fixing $(t,r)$ and then using the process

$$\exp\left\{-\int_t^s r(u)du\right\} F^T(s,r(s)).$$

(16.20)

If we apply the Itô formula to (16.20) and use the fact that $F^T$ satisfies the term structure equation then, by using exactly the same technique as in Section 4.5, we obtain the following stochastic representation formula.

**Proposition 16.3 (Risk neutral valuation)** *Bond prices are given by the formula $p(t,T) = F(t,r(t);T)$ where*

$$F(t,r;T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds}\right]$$

(16.21)

*Here the martingale measure Q and the subscripts t,r denote that the expectation shall be taken given the following dynamics for the short rate.*

$$dr(s) = \{\mu - \lambda\sigma\} ds + \sigma dW(s),$$

(16.22)

$$r(t) = r.$$

(16.23)

The formula (16.21) has the usual natural economic interpretation, which is most easily seen if we write it as

$$F(t,r;T) = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds} \times 1\right]$$

(16.24)

We see that the value of a $T$-bond at time $t$ is given as the expected value of the final payoff of one pound, discounted to present value. The deflator used is the natural one, namely $\exp\{-\int_t^T r(s)ds\}$, but we observe that the expectation is not to be taken using the underlying objective probability measure *P*. Instead we must, as usual, use the martingale measure *Q* and we see that we have different martingale measures for different choices of λ.

The main difference between the present situation and the Black-Scholes setting is that in the Black-Scholes model the martingale measure is uniquely determined. It can be shown (see Chapters 9 and 10) that the uniqueness of the martingale measure is due to the fact that the Black-Scholes model is complete. In the present case our exogenously given market is not complete, so bond prices will not be uniquely determined by the given (*P*-)dynamics of the short rate *r*. To express this fact more precisely, the various bond prices will be determined **partly** by the *P*-dynamics of the short rate of interest, and **partly** by market forces. The fact that there are different possible choices of λ simply means that there are different conceivable bond markets all of which are consistent with the given *r*-dynamics. Precisely which set of bond price processes will be realized by an actual market will depend on the relations between supply and demand for bonds in this particular market, and these factors are in their turn determined by such things as the forms of risk aversion possessed by the various agents on the market. In particular this means that if we make an *ad hoc* choice of λ (e.g. such as $\lambda = 0$) then we have implicitly made an assumption concerning the aggregate risk aversion on the market.

We can also turn the argument around and say that when the market has determined the dynamics of **one** bond price process, say with maturity *T*, then the market has indirectly specified λ by eqn (16.18). When λ is thus determined, all other bond prices will be determined by the term structure equation. Expressed in another way: all bond prices will be determined **in terms of** the basic *T*-bond and the short rate of interest. Again we see that arbitrage pricing always is a case of determining prices of derivatives **in terms of** some a priori given price processes.

There remains one important and natural question, namely how we ought to choose λ in a concrete case. This question will be treated in some detail, in Section 17.2, and the moral is that we must go to the actual market and, by using market data, infer the market's choice of λ.

The bonds treated above are of course contingent claims of a particularly simple type; they are deterministic. Let us close this section by looking at a more general type of contingent *T*-claim of the form

$$\chi = \Phi(r(T)), \tag{16.25}$$

where Φ is some real valued function. Using the same type of arguments as above it is easy to see that we have the following result.

**Proposition 16.4 (General term structure equation)** *Let $\chi$ be a contingent T-claim of the form $\chi = \Phi(r(T))$. In an arbitrage free market the price* $\Pi\ (t; \Phi)$ *will be given as*

$$\Pi(t, \Phi) = F(t, r(t)),$$

*where F solves the boundary value problem*

(16.26)

$$\begin{cases} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF & = 0, \\ F(T, r) & = \Phi(r). \end{cases}$$

(16.27)

*Furthermore F has the stochastic representation*

$$F(t, r; T) = E_{t,r}^{Q}\left[\exp\left\{-\int_t^T r(s)ds\right\} \times \Phi(r(T))\right],$$

(16.28)

*where the martingale measure Q and the subscripts t,r denote that the expectation shall be taken using the following dynamics.*

$$dr(s) = \{\mu - \lambda\sigma\} ds + \sigma dW(s),$$

(16.29)

$$r(t) = r.$$

(16.30)

## 16.3 Exercises

**Exercise 16.1** We take as given an interest rate model with the following $P$-dynamics for the short rate.

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\overline{W}(t).$$

Now consider a $T$-claim of the form $\chi = \Phi(r(T))$ with corresponding price process $\Pi(t)$.

(a) Show that, under any martingale measure $Q$, the price process $\Pi(t)$ has a local rate of return equal to the short rate of interest. In other words, show that the stochastic differential of $\Pi(t)$ is of the form

$$d\Pi(t) = r(t)\Pi(t)dt + \sigma_\Pi \Pi(t)dW(t).$$

(b) Show that the normalized price process

$$Z(t) = \frac{\Pi(t)}{B(t)}$$

is a $Q$-martingale.

**Exercise 16.2** The object of this exercise is to connect the forward rates defined in Chapter 15 to the framework above.

(a) Assuming that we are allowed to differentiate under the expectation sign, show that

$$f(t, T) = \frac{E_{t,r(t)}^{Q}\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E_{t,r(t)}^{Q}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]}.$$

(b) Check that indeed $r(t) = f(t,t)$.

**Exercise 16.3 (Swap a fixed rate vs. a short rate)** Consider the following version of an interest rate swap. The contract is made between two parties, A and B, and the payments are made as follows.

- A (hypothetically) invests the principal amount $K$ at time 0 and lets it grow at a fixed rate of interest $R$ (to be determined below) over the time interval $[0,T]$.
- At time $T$ the principal will have grown to $K_A$ SEK. A will then subtract the principal amount and pay the surplus $K - K_A$ to B (at time $T$).
- B (hypothetically) invests the principal at the stochastic short rate of interest over the interval $[0,T]$.
- At time $T$ the principal will have grown to $K_B$ SEK. B will then subtract the principal amount and pay the surplus $K - K_B$ to A (at time $T$).

The **swap rate** for this contract is now defined as the value, $R$, of the fixed rate which gives this contract the value zero at $t = 0$. Your task is to compute the swap rate.

**Exercise 16.4 (Forward contract)** Consider a model with a stochastic rate of interest. Fix a $T$-claim $\chi$ of the form $\chi = \Phi(r(T))$, and fix a point in time $t$, where $t < T$. From Proposition 16.4 we can in principle compute the arbitrage free price for $\chi$ if we pay at time $t$. We may also consider a **forward contract** (see Section 6.6.1) on $\chi$ **contracted at** $t$. This contract works as follows, where we assume that you are the buyer of the contract.

- At time $T$ you obtain the amount $\chi$ SEK.
- At time $T$ you pay the amount $K$ SEK.
- The amount $K$ is determined at $t$.

The **forward price for** $\chi$ **contracted at** $t$ is defined as the value of $K$ which gives the entire contract the value zero at time $t$. Give a formula for the forward price.

## 16.4 Notes

The exposition in this chapter is standard. For further information, see the notes at the end of the next chapter.

# 17 Martingale Models for the Short Rate

## 17.1 Q-Dynamics

Let us again study an interest rate model where the *P*-dynamics of the short rate of interest are given by

$$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))d\overline{W}. \tag{17.1}$$

As we saw in the previous chapter, the term structure (i.e. the family of bond price processes) will, together with all other derivatives, be completely determined by the general term structure equation

$$\begin{cases} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T,r) &= \Phi(r), \end{cases} \tag{17.2}$$

as soon as we have specified the following objects.

- The drift term $\mu$.
- The diffusion term $\sigma$.
- The market price of risk $\lambda$.

Consider for a moment $\sigma$ to be given a priori. Then it is clear from (17.2) that it is irrelevant exactly how we specify $\mu$ and $\lambda$ *per se*. The object, apart from $\sigma$, that really determines the term structure (and all other derivatives) is the term $\mu - \lambda \sigma$ in eqn (17.2). Now, from Proposition 16.4 we recall that the term $\mu - \lambda \sigma$ is precisely the drift term of the short rate of interest under the martingale measure *Q*. This fact is so important that we stress it again.

*Result 17.1.1 The term structure, as well as the prices of all other interest rate derivatives, are completely determined by specifying the* **r-dynamics under the martingale measure** *Q*.

Instead of specifying $\mu$ and $\lambda$ under the objective probability measure *P* we will henceforth specify the dynamics of the short rate *r* directly under the martingale meaure *Q*. This procedure is known as **martingale modeling** , and the typical assumption will thus be that *r* under *Q* has dynamics given by

$$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))dW(t), \tag{17.3}$$

where $\mu$ and $\sigma$ are given functions. From now on the letter $\mu$ will thus always denote the drift term of the short rate of interest under the martingale measure *Q*.

In the literature there are a large number of proposals on how to specify the $Q$-dynamics for $r$. We present a (far from complete) list of the most popular models. If a parameter is time dependent this is written out explicitly. Otherwise all parameters are constant.

1. Vasiček

$$dr = (b - ar)dt + \sigma dW, \ (a > 0), \tag{17.4}$$

2. Cox-Ingersoll-Ross (CIR)

$$dr = a(b - r)dt + \sigma\sqrt{r}dW, \tag{17.5}$$

3. Dothan

$$dr = ardt + \sigma rdW, \tag{17.6}$$

4. Black-Derman-Toy

$$dr = \Theta(t)rdt + \sigma(t)rdW, \tag{17.7}$$

5. Ho-Lee

$$dr = \Theta(t)dt + \sigma dW, \tag{17.8}$$

6. Hull-White (extended Vasiček)

$$dr = (\Theta(t) - a(t)r)dt + \sigma(t)dW, \ (a(t) > 0), \tag{17.9}$$

7. Hull-White (extended CIR)

$$dr = (\Theta(t) - a(t)r)dt + \sigma(t)\sqrt{r}dW. \ (a(t) > 0). \tag{17.10}$$

## 17.2 Inversion of the Yield Curve

Let us now address the question of how we will estimate the various model parameters in the martingale models above. To take a specific case, assume that we have decided to use the Vasiček model. Then we have to get values for $a$, $b$, and $\sigma$ in some way, and a natural procedure would be to look in some textbook dealing with parameter estimation for SDEs. This procedure, however, is unfortunately completely nonsensical and the reason is as follows.

We have chosen to model our $r$-process by giving the $Q$-dynamics, which means that $a$, $b$ and $\sigma$ are the parameters which hold under the martingale measure $Q$. When we make observations in the real world we are **not** observing $r$ under the martingale measure $Q$, but under the objective measure $P$. This means that if we apply standard statistical procedures to our observed data we will not get our $Q$-parameters. What we get instead is pure nonsense.

This looks extremely disturbing but the situation is not hopeless. It is in fact possible to show that the diffusion term is the same under $P$ and under $Q$, so "in principle" it may be possible to estimate diffusion parameters using $P$-data. (The reader familiar with martingale theory will at this point recall that a Girsanov transformation will only affect the drift term of a diffusion but not the diffusion term.)

When it comes to the estimation of parameters affecting the drift term of $r$ we have to use completely different methods.

From Section 10.5 we recall the following moral.

**Question:**

**Who chooses the martingale measure?**

**Answer:**

**The market!**

Thus, in order to obtain information about the $Q$-drift parameters we have to collect price information from the market, and the typical approach is that of **inverting the yield curve** which works as follows. (See the more detailed discussion in Section 10.5.)

- Choose a particular model involving one or several parameters. Let us denote the entire parameter vector by $\alpha$. Thus we write the $r$-dynamics (under $Q$) as

$$dr(t) = \mu(t, r(t); \alpha)dt + \sigma(t, r(t); \alpha)dW(t). \tag{17.11}$$

- Solve, for every conceivable time of maturity $T$, the term structure equation

$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\ F^T(T, r) &= 1. \end{cases} \tag{17.12}$$

  In this way we have computed the theoretical term structure as

$$p(t, T; \alpha) = F^T(t, r; \alpha).$$

  Note that the form of the term structure will depend upon our choice of parameter vector. We have not made this choice yet.
- Collect price data from the bond market. In particular we may today (i.e. at $t = 0$) observe $p(0, T)$ for all values of $T$. Denote this **empirical term structure** by $\{p^*(0, T); T \geq 0\}$.
- Now choose the parameter vector $\alpha$ in such a way that the theoretical curve $\{p(0, T; \alpha); T \geq 0\}$ fits the empirical curve $\{p^*(0, T); T \geq 0\}$ as well as possible (according to some objective function). This gives us our estimated parameter vector $\alpha^*$.

- Insert $\alpha^*$ into $\mu$ and $\sigma$. Now we have pinned down exactly which martingale measure we are working with. Let us denote the result of inserting $\alpha^*$ into $\mu$ and $\sigma$ by $\mu^*$ and $\sigma^*$ respectively.
- We have now pinned down our martingale measure *Q*, and we can go on to compute prices of interest rate derivatives, like, say, $\chi = \Gamma(r(T))$. The price process is then given by $\Pi(t;\Gamma) = G(t, r(t))$ where *G* solves the term structure equation

$$\begin{cases} G_t + \mu^* G_r + \frac{1}{2}[\sigma^*]^2 G_{rr} - rG & = 0, \\ G(T,r) & = \Gamma(r). \end{cases} \tag{17.13}$$

If the above program is to be carried out within reasonable time limits it is of course of great importance that the PDEs involved are easy to solve. It turns out that some of the models above are much easier to deal with analytically than the others, and this leads us to the subject of so called **affine term structures** .

# 17.3 Affine Term Structures

## 17.3.1 Definition and Existence

**Definition 17.1** *If the term structure* $\{p(t, T); 0 \le t \le T, T > 0\}$ *has the form*

$$p(t,T) = F(t,r(t); T), \tag{17.14}$$

*where F has the form*

$$F(t,r; T) = e^{A(t,T) - B(t,T)r}, \tag{17.15}$$

*and where A and B are deterministic functions, then the model is said to possess an* **affine term structure** *(ATS).*

The functions *A* and *B* above are functions of the two real variables *t* and *T*, but conceptually it is easier to think of *A* and *B* as being functions of *t*, while *T* serves as a parameter. It turns out that the existence of an affine term structure is extremely pleasing from an analytical and a computational point of view, so it is of considerable interest to understand when such a structure appears. In particular we would like to answer the following question.

- For which choices of $\mu$ and $\sigma$ in the *Q*-dynamics for *r* do we get an affine term structure?

We will try to give at least a partial answer to this question, and we start by investigating some of the implications of an affine term structure. Assume then that we have the *Q*-dynamics

$$dr(t) = \mu(t,r(t))dt + \sigma(t,r(t))dW(t), \tag{17.16}$$

and assume that this model actually possesses an ATS. In other words we assume that the bond prices have the form (17.15) above. Using (17.15) we may easily compute the various partial derivatives of $F$, and since $F$ must solve the term structure equation (16.19), we thus obtain

$$A_t(t,T) - \{1 + B_t(t,T)\} r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{17.17}$$

The boundary value $F(T, r;T) \equiv 1$ implies

$$\begin{cases} A(T,T) & = 0, \\ B(T,T) & = 0. \end{cases} \tag{17.18}$$

Equation (17.17) gives us the relations which must hold between $A$, $B$, $\mu$ and $\sigma$ in order for an ATS to exist, and for a certain choice of $\mu$ and $\sigma$ there may or may not exist functions $A$ and $B$ such that (17.17) is satisfied. Our immediate task is thus to give conditions on $\mu$ and $\sigma$ which guarantee the existence of functions $A$ and $B$ solving (17.17). Generally speaking this is a fairly complex question, but we may give a very nice partial answer. We observe that if $\mu$ and $\sigma^2$ are both **affine** (i.e. linear plus a constant) functions of $r$, with possibly time dependent coefficients, then eqn (17.17) becomes a separable differential equation for the unknown functions $A$ and $B$.

Assume thus that $\mu$ and $\sigma$ have the form

$$\begin{cases} \mu(t,r) & = \alpha(t)r + \beta(t), \\ \sigma(t,r) & = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{17.19}$$

Then, after collecting terms, (17.17) transforms into

$$\begin{aligned} &A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) \\ &- \left\{1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T)\right\} r = 0. \end{aligned} \tag{17.20}$$

This equation holds for all $t$, $T$ and $r$, so let us consider it for a fixed choice of $T$ and $t$. Since the equation holds for all values of $r$ the coefficient of $r$ must be equal to zero. Thus we have the equation

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1. \tag{17.21}$$

Since the $r$-term in (17.20) is zero we see that the other term must also vanish, giving us the equation

$$A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T). \tag{17.22}$$

We may thus formulate our main result.

**Proposition 17.2 (Affine term structure)** *Assume that* μ *and* σ *are of the form*

$$\begin{cases} \mu(t,r) &= \alpha(t)r + \beta(t), \\ \sigma(t,r) &= \sqrt{\gamma(t)r + \delta(t)}. \end{cases}$$

(17.23)

*Then the model admits an ATS of the form* (17.15), *where A and B satisfy the system*

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) &= -1, \\ B(T,T) &= 0. \end{cases}$$

(17.24)

$$\begin{cases} A_t(t,T) &= \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ A(T,T) &= 0. \end{cases}$$

(17.25)

We note that eqn (17.24) is a Ricatti equation for the determination of *B* which does not involve *A*. Having solved eqn (17.24) we may then insert the solution *B* into eqn (17.25) and simply integrate in order to obtain *A*.

An interesting question is if it is only for an affine choice of μ and $\sigma^2$ that we get an ATS. This is not generally the case, but it can fairly easily be shown that if we demand that μ and $\sigma^2$ are time independent, then a necessary Condition for the existence of an ATS is that μ and $\sigma^2$ are affine. Looking at the list of models in the previous section we see that all models except the Dothan and the Black-Derman-Toy models have an ATS.

## 17.3.2 A Probabilistic Discussion

There are good probabilistic reasons why some of the models in our list are easier to handle than others. We see that the models of Vasiček, Ho-Lee and Hull-White (extended Vasiček) all describe the short rate using a **linear** SDE. Such SDEs are easy to solve and the corresponding *r*-processes can be shown to be normally distributed. Now, bond prices are given by expressions like

$$p(0,T) = E\left[\exp\left\{-\int_0^T r(s)ds\right\}\right],$$

(17.26)

and the normal property of *r* is inherited by the integral $\int_0^T r(s)ds$ (an integral is just a sum). Thus we see that the computation of bond prices for a model with a normally distributed short rate boils down to the easy problem of computing the expected value of a log-normal stochastic variable. This purely probabilistic program can in fact be carried out for all the linear models above (the interested reader is invited to do this), but it turns out that from a computational point of view it is easier to solve the system of equations (17.24)–(17.25).

In contrast with the linear models above, consider for a moment the Dothan model. This model for the short rate is the same as the Black-Scholes model for the underlying stock, so one is easily led to believe that computationally this is the nicest model conceivable. This is, however, not the case. For the Dothan model the short rate will be log-normally distributed, which means that in order to compute bond prices we are faced with determining the distribution of an integral $\int_0^T r(s)ds$ of log-normal stochastic variables. It is, however, a sad fact that a sum (or an integral) of log-normally distributed variables is a particularly nasty object, so this model leads to great computational problems. It also has the unreasonable property that the expected value of the money account equals plus infinity.

As for the Cox-Ingersoll-Ross model and the Hull-White extension, these models for the short rate are roughly obtained by taking the square of the solution of a linear SDE, and can thus be handled analytically (see the exercises for a simplified example). They are, however, quite a bit messier to deal with than the normally distributed models. See the notes.

From a computational point of view there is thus a lot to be said in favour of a linear SDE describing the short rate. The price we have to pay for these models is again the Gaussian property. Since the short rate will be normally distributed this means that for every $t$ there is a positive probability that $r(t)$ is negative, and this is unreasonable from an economic point of view. For the Dothan model on the other hand, the short rate is log-normal and thus positive with probability 1. It is also possible to show that the Cox-Ingersoll-Ross model will produce a strictly positive short rate process. See Rogers (1995) for a discussion on these problems.

We end this section with a comment on the procedure of calibrating the model to data described in the previous section. If we want a complete fit between the theoretical and the observed bond prices this calibration procedure is formally that of solving the system of equations

$$p(0,T;\alpha) = p^*(0,T) \quad \text{for all } T > 0. \tag{17.27}$$

We observe that this is an infinite dimensional system of equations (one equation for each $T$) with $\alpha$ as the unknown, so if we work with a model containing a finite parameter vector $\alpha$ (like the Vasiček model) there is no hope of obtaining a perfect fit. Now, one of the main goals of interest rate theory is to compute prices of various derivatives, like, for example, bond options, and it is well known that the price of a derivative can be very sensitive with respect to the price of the underlying asset. For bond options the underlying asset is a bond, and it is thus disturbing if we have a model for derivative pricing which is not even able to correctly price the underlying asset.

This leads to a natural demand for models which **can** be made to fit the observed bond data completely, and this is the reason why the Hull-White model has become so popular. In this model (and related ones) we introduce an infinite dimensional parameter vector $\alpha$ by letting some or all parameters be time dependent. Whether it is possible to actually solve the system (17.27) for a concrete model such as the Hull-White extension of the Vasiček model, and how this is to be done in detail, is of course not clear a priori but has to be dealt with in a deeper study. We carry out this study for the Hull-White model in the next section.

It should, however, be noted that the introduction of an infinite parameter, in order to fit the entire initial term structure, has its dangers in terms of numerical instability of the parameter estimates.

There is also a completely different approach to the problem of obtaining a perfect fit between today's theoretical bond prices and today's observed bond prices. This is the Heath-Jarrow-Morton approach which roughly takes the observed term structure as an initial Condition for the forward rate curve, thus automatically obtaining a perfect fit. This model will be studied in the next chapter.

# 17.4 Some Standard Models

In this section we will apply the ATS theory above, in order to study the most common affine one factor models.

## 17.4.1 The Vasiček Model

To illustrate the technique we now compute the term structure for the Vasiček model

$$dr = (b - ar)dt + \sigma dW.$$

(17.28)

Before starting the computations we note that this model has the property of being **mean reverting** (under $Q$) in the sense that it will tend to revert to the mean level $b/a$. The equations (17.24)–(17.25) become

$$\begin{cases} B_t(t,T) - aB(t,T) &= -1, \\ B(T,T) &= 0. \end{cases}$$

(17.29)

$$\begin{cases} A_t(t,T) &= bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) &= 0. \end{cases}$$

(17.30)

Equation (17.29) is, for each fixed $T$, a simple linear ODE in the $t$-variable. It can easily be solved as

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}.$$

(17.31)

Integrating eqn (17.30) we obtain

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - b\int_t^T B(s,T)ds,$$

(17.32)

and, substituting the expression for $B$ above, we obtain the following result.

**Proposition 17.3 (The Vasiček term structure)** *In the Vasiček model, bond prices are given by*

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)},$$

*where*

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\},$$

$$A(t,T) = \frac{\{B(t,T) - T + t\}\,(ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

For the Vasiček model, there is also an explicit formula for European bond options. See Proposition 17.9 below.

## 17.4.2 The Ho-Lee Model

For the Ho-Lee model the ATS equations become

$$\begin{cases} B_t(t,T) = -1, \\ B(T,T) = 0. \end{cases}$$

$$\begin{cases} A_t(t,T) = \Theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases}$$

These are easily solved as

$$B(t,T) = T - t,$$

$$A(t,T) = \int_t^T \Theta(s)(s-T)ds + \frac{\sigma^2}{2}\cdot\frac{(T-t)^3}{3}.$$

It now remains to choose $\Theta$ such that the theoretical bond prices, at $t = 0$, fit the observed initial term structure $\{p^*(0, T); T \geq 0\}$. We thus want to find $\Theta$ such that $p(0, T) = p^*(0, T)$ for all $T \geq 0$. This is left as an exercise, and the solution is given by

$$\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t,$$

where $f(0, t)$ denotes the observed forward rates. Plugging this expression into the ATS gives us the following bond prices.

**Proposition 17.4 (The Ho-Lee term structure)** *For the Ho-Lee model, the bond prices are given by*

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\left\{(T-t)f(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)\right\}$$

For completeness we also give the pricing formula for a European call on an underlying bond. We will not derive this result by solving the pricing PDE (this is in fact very hard), but instead we refer the reader to Chapter 19 where we will present a rather general option pricing formula (Proposition 19.14). It is then an easy exercise to obtain the result below as a special case.

**Proposition 17.5 (Bond options)** *For the Ho-Lee model, the price at t, of a European call option with strike price K and exercise date T, on an underlying S-bond, we have the following pricing formula.*

$$c(t,T,K,S) = p(t,S)N(d) - p(t,T)\cdot K \cdot N(d-\sigma_p),$$

*where*

(17.33)

$$d = \frac{1}{\sigma_p}\log\left\{\frac{p(t,S)}{p(t,T)K}\right\} + \frac{1}{2}\sigma_p,$$

$$\sigma_p = \sigma(S-T)\sqrt{T}. \tag{17.34}$$

(17.35)

## 17.4.3 The CIR Model

The CIR model is much more difficult to handle than the Vasiček model, since we have to solve a Riccati equation. We cite the following result.

**Proposition 17.6 (The CIR term structure)** *The term structure for the CIR model is given by*

$$F^T(t,r) = A_0(T-t)e^{-B(T-t)r},$$

*where*

$$B(x) = \frac{2(e^{\gamma x}-1)}{(\gamma+a)(e^{\gamma x}-1)+2\gamma},$$

$$A_0(x) = \left[\frac{2\gamma e^{(a+\gamma)(x/2)}}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}\right]^{2ab/\sigma^2},$$

*and*

$$\gamma = \sqrt{a^2+2\sigma^2}.$$

It is possible to obtain closed form expressions for European call options on zero coupon bonds within the CIR framework. Since these formulas are rather complicated, we refer the reader to Cox-Ingersoll-Ross (1985b).

## 17.4.4 The Hull-White Model

In this section we will make a fairly detailed study of a simplified version of the Hull-White extension of the Vasiček model. The $Q$-dynamics of the short rate are given by

$$dr = \{\Theta(t) - ar\}dt + \sigma dW(t),$$

(17.36)

where $a$ and $\sigma$ are constants while $\Theta$ is a deterministic function of time. In this model we typically choose $a$ and $\sigma$ in order to obtain a nice volatility structure whereas $\Theta$ is chosen in order to fit the theoretical bond prices $\{p(0, T); T > 0\}$ to the observed curve $\{p^*(0, T); T > 0\}$.

We have an affine structure so by Proposition 17.2 bond prices are given by

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

where $A$ and $B$ solve

(17.37)

$$\begin{cases} B_t(t,T) &= aB(t,T) - 1, \\ B(T,T) &= 0. \end{cases}$$

$$\begin{cases} A_t(t,T) &= \Theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) &= 0. \end{cases} \tag{17.38}$$

The solutions to these equations are given by (17.39)

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\},$$

$$A(t,T) = \int_t^T \left\{\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\right\}ds. \tag{17.40}$$

(17.41)

Now we want to fit the theoretical prices above to the observed prices and it is convenient to do this using the forward rates. Since there is a one-to-one correspondence (see Lemma 15.4) between forward rates and bond prices, we may just as well fit the theoretical forward rate curve $\{f(0,T);\ T > 0\}$ to the observed curve $\left\{f^*(0,T);\ T > 0\right\}$, where of course $f$ is defined by $f^*(t,T) = -\frac{\partial \log p^*(t,T)}{\partial T}$. In any affine model the forward rates are given by

$$f(0,T) = B_T(0,T)r(0) - A_T(0,T),$$

which, after inserting (17.40)–(17.41), becomes

(17.42)

$$f(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\Theta(s)ds - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2.$$

(17.43)

Given an observed forward rate structure $f$ our problem is to find a function $\Theta$ which solves the equation

$$f^*(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\Theta(s)ds - \frac{\sigma^2}{2a^2}(1 - e^{-aT})^2,\ \forall T > 0.$$

One way of solving (17.44) is to write it as

(17.44)

$$f^*(0,T) = x(T) - g(T),$$

where $x$ and $g$ are defined by

(17.45)

$$\begin{cases} \dot{x} &= -ax(t) + \Theta(t), \\ x(0) &= r(0), \end{cases}$$

$$g(t) = \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 = \frac{\sigma^2}{2}B^2(0,t).$$

We now have

(17.46)

$$\begin{aligned} \Theta(T) &= \dot{x}(T) + ax(T) = f^*_T(0,T) + \dot{g}(T) + ax(T) \\ &= f^*_T(0,T) + \dot{g}(T) + a\left\{f^*(0,T) - g(T)\right\}, \end{aligned}$$

(17.47)

so we have in fact proved the following result.

**Lemma 17.7** *Fix an arbitrary bond curve* $\{p^*(0, T); T > 0\}$, *subject only to the Condition that* $p^*(0, T)$ *is twice differentiable w.r.t.* $T$. *Choosing* $\Theta$ *according to* (*17.47*) *will then produce a term structure* $\{p(0, T); T > 0\}$ *such that* $p(0, T) = p^*(0, T)$ *for all* $T > 0$.

By choosing $\Theta$ according to (17.47) we have, for a fixed choice of $a$ and $\sigma$, determined our martingale measure. Now we would like to compute the theoretical bond prices under this martingale measure, and in order to do this we have to substitute our choice of $\Theta$ into eqn (17.41). Then we perform the integration and substitute the result as well as eqn (17.40) into eqn (17.37). This leads to some exceedingly boring calculations which (of course) are left to the reader. The result is as follows.

**Proposition 17.8 (The Hull-White term structure)** *Consider the Hull-White model with a and* $\sigma$ *fixed. Having inverted the yield curve by choosing* $\Theta$ *according to* (*17.47*) *we obtain the bond prices as*

$$p(t,T) = \frac{p(0,T)}{p(0,t)} \exp\left\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)\right\}, \tag{17.48}$$

*where B is given by* (*17.40*).

We end this section by giving, for the Hull-White, as well as for the Vasiček model, the pricing formula for a European call option with time of maturity $T$ and strike price $K$ on an $S$-bond, where of course $T < S$. We denote this price by $c(t, T, K, S)$. At the present stage the reader is not encouraged to derive the formula below. In Chapter 19 we will instead present a technique which will greatly simplify computations of this kind, and the formula will be derived with relative ease in Section 19.6 (see Proposition 19.16). Note that the bond prices $p(t, T)$ and $p(t, S)$ below do not have to be computed at time $t$, since they can be observed directly on the market.

**Proposition 17.9 (Bond options)** *Using notations as above we have, both for the Hull-White and the Vasiček models, the following bond option formula.*

$$c(t,T,K,S) = p(t,S)N(d) - p(t,T)\cdot K \cdot N(d-\sigma_p), \tag{17.49}$$

*where*

$$d = \frac{1}{\sigma_p}\log\left\{\frac{p(t,S)}{p(t,T)K}\right\} + \frac{1}{2}\sigma_p, \tag{17.50}$$

$$\sigma_p = \frac{1}{a}\left\{1-e^{-a(S-T)}\right\}\cdot\sqrt{\frac{\sigma^2}{2a}\left\{1-e^{-2a(T-t)}\right\}}. \tag{17.51}$$

# 17.5 Exercises

**Exercise 17.1** Consider the Vasiček model, where we always assume that $a > 0$.

(a) Solve the Vasiček SDE explicitly, and determine the distribution of $r(t)$. **Hint:** The distribution is Gaussian (why?), so it is enough to compute the expected value and the variance.

(b) As $t \to \infty$, the distribution of $r(t)$ tends to a limiting distribution. Show that this is the Gaussian distribution $N[b/a, \sigma/\sqrt{2a}]$. Thus we see that, in the limit, $r$ will indeed oscillate around its mean reversion level $b/a$.

(c) Now assume that $r(0)$ is a stochastic variable, independent of the Wiener process $W$, and by definition having the Gaussian distribution obtained in (b). Show that this implies that $r(t)$ has the limit distribution in (b), for all values of $t$. Thus we have found the stationary distribution for the Vasiček model.

(d) Check that the density function of the limit distribution solves the time invariant Fokker-Planck equation, i.e. the Fokker-Planck equation with the $\frac{\partial}{\partial t}$-term equal to zero.

**Exercise 17.2** Show directly that the Vasiček model has an affine term structure without using the methodology of Proposition 17.2. Instead use the characterization of $p(t, T)$ as an expected value, insert the solution of the SDE for $r$, and look at the structure obtained.

**Exercise 17.3** Try to carry out the program outlined above for the Dothan model and convince yourself that you will only get a mess.

**Exercise 17.4** Show that for the Dothan model you have $E^Q[B(t)] = \infty$.

**Exercise 17.5** Consider the Ho-Lee model

$$dr = \Theta(t)dt + \sigma dW(t).$$

Assume that the observed bond prices at $t = 0$ are given by $\{p^*(0, T); t \geq 0\}$. Assume furthermore that the constant $\sigma$ is given. Show that this model can be fitted exactly to today's observed bond prices with $\Theta$ as

$$\Theta(t) = \frac{\partial f^*}{\partial T}(0,t) + \sigma^2 t,$$

where $f^*$ denotes the observed forward rates. (The observed bond price curve is assumed to be smooth.)

**Hint:** Use the affine term strucuture, and fit forward rates rather than bond prices (this is logically equivalent).

**Exercise 17.6** Use the result of the previous exercise in order to derive the bond price formula in Proposition 17.4.

**Exercise 17.7** It is often considered reasonable to demand that a forward rate curve always has an horizontal asymptote, i.e. that $\lim_{T\to\infty} f(t, T)$ exists for all $t$. (The limit will obviously depend upon $t$ and $r(t)$). The object of this exercise is to show that the Ho-Lee model is not consistent with such a demand.

(a) Compute the explicit formula for the forward rate curve $f(t, T)$ for the Ho-Lee model (fitted to the initial term structure).
(b) Now assume that the initial term structure indeed has a horizontal asymptote, i.e. that $\lim_{T\to\infty} f(0, T)$ exists. Show that this property is not respected by the Ho-Lee model, by fixing an arbitrary time $t$, and showing that $f(t, T)$ will be asymptotically linear in $T$.

**Exercise 17.8** The object of this exercise is to indicate why the CIR model is connected to squares of linear diffusions. Let $Y$ be given as the solution to the following SDE.

$$dY = \left(2aY + \sigma^2\right)dt + 2\sigma\sqrt{Y}dW,\ Y(0) = y_0.$$

Define the process $Z$ by $Z(t) = \sqrt{Y(t)}$. It turns out that $Z$ satisfies a stochastic differential equation. Which?

# 17.6 Notes

Basic papers on short rate models are Vasiček (1977), Hull and White (1990), Ho and Lee (1986), Cox *et al.* (1985b), Dothan (1978), and Black *et al.* (1990). For extensions and notes on the affine term structure theory, see Duffie and Kan (1996). An extensive analysis of the linear quadratic structure of the CIR model can be found in Magshoodi (1996). The bond option formula for the Vasiček model was first derived by Jamshidian (1989). For examples of two-factor models see Brennan and Schwartz (1979), (1982), and Longstaff and Schwartz (1992). Rogers (1997) shows how it is possible to generate a wide class of short rate models by modelling the state price density directly under $P$ and using resolvents. A completely different approach to interest rate theory is given in Platen (1996) where the short rate is derived as a consequence of an entropy related principle. See also Platen and Rebolledo (1995). For an overview of interest rate theory see Björk (1997).

# 18 Forward Rate Models

## 18.1 The Heath-Jarrow-Morton Framework

Up to this point we have studied interest models where the short rate $r$ is the only explanatory variable. The main advantages with such models are as follows.

- Specifying $r$ as the solution of an SDE allows us to use Markov process theory, so we may work within a PDE framework.
- In particular it is often possible to obtain analytical formulas for bond prices and derivatives.

The main drawbacks of short rate models are as follows.

- From an economic point of view it seems unreasonable to assume that the entire money market is governed by only one explanatory variable.
- It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.
- As the short rate model becomes more realistic, the inversion of the yield curve described above becomes increasingly more difficult.

These, and other considerations, have led various authors to propose models which use more than one state variable. One obvious Idea would, for example, be to present an a priori model for the short rate as well as for some long rate, and one could of course also model one or several intermediary interest rates. The method proposed by Heath-Jarrow-Morton is at the far end of this spectrum—they choose the entire forward rate curve as their (infinite dimensional) state variable.

We now turn to the specification of the Heath-Jarrow-Morton (HJM) framework. We start by specifying everything under a given objective measure $P$.

**Assumption 18.1.1** *We assume that, for every fixed $T > 0$, the forward rate $f(\cdot, T)$ has a stochastic differential which under the objective measure $P$ is given by*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\overline{W}(t), \tag{18.1}$$

$$f(0,T) = f^*(0,T), \tag{18.2}$$

*where $\overline{W}$ is a (d-dimensional) P-Wiener process whereas $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes.*

Note that conceptually eqn (18.1) is one stochastic differential in the $t$-variable for each fixed choice of $T$. The index $T$ thus only serves as a "mark" or "parameter" in order to indicate which maturity we are looking at. Also note that we use the observed forward rated curve $\{f(0, T); T \geq 0\}$ as the initial Condition. This will automatically give us a perfect fit between observed and theoretical bond prices at $t = 0$, thus relieving us of the task of inverting the yield curve.

**Remark 18.1.1** It is important to observe that the HJM approach to interest rates is not a proposal of a specific **model** , like, for example, the Vasiček model. It is instead a **framework** to be used for analyzing interest rate models. Every short rate model can be equivalently formulated in forward rate terms, and for every forward rate model, the arbitrage free price of a contingent $T$-claim $\chi$ will still be given by the pricing formula

$$\Pi(0; \chi) = E^{Q}\left[\exp\left\{\int_0^T r(s)ds\right\} \cdot \chi\right],$$

where the short rate as usual is given by $r(s) = f(s, s)$.

Suppose now that we have specified α, σ and $\{f(0, T); T \geq 0\}$. Then we have specified the entire forward rate structure and thus, by the relation

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}, \tag{18.3}$$

we have in fact specified the entire term structure $\{p(t, T); T > 0, 0 \leq t \leq T\}$. Since we have $d$ sources of randomness (one for every Wiener process), and an infinite number of traded assets (one bond for each maturity $T$), we run a clear risk of having introduced arbitrage possibilities into the bond market. The first question we pose is thus very natural: How must the processes α and σ be related in order that the induced system of bond prices admits no arbitrage possibilities? The answer is given by the HJM drift Condition below.

**Theorem 18.1 (HJM drift Condition)** *Assume that the family of forward rates is given by (18.1) and that the induced bond market is arbitrage free. Then there exists a d-dimensional column-vector process*

$$\lambda(t) = [\lambda_1(t),\ldots,\lambda_d(t)]'$$

*with the property that for all $T \geq 0$ and for all $t \leq T$, we have*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds - \sigma(t,T)\lambda(t). \tag{18.4}$$

*In these formulas ' denotes transpose.*

**Proof** From Proposition 15.5 we have the bond dynamics

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\left\|S(t,T)\right\|^2\right\}dt + p(t,T)S(t,T)d\bar{W}(t), \tag{18.5}$$

where

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds. \end{cases}$$

The risk premium for the $T$-bond is thus given by

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2, \tag{18.6}$$

and, applying Result 10.5.1, we conclude the existence of a $d$-dimensional column-vector process $\lambda$ such that

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 = \sum_{i=1}^{d} S_i(t,T)\lambda_i(t).$$

Taking the $T$-derivative of this equation gives us eqn (18.4).

## 18.2 Martingale Modeling

We now turn to the question of martingale modeling, and thus assume that the forward rates are specified directly under a martingale measure $Q$ as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{18.7}$$

$$f(0,T) = f^*(0,T), \tag{18.8}$$

where $W$ is a ($d$-dimensional) $Q$-Wiener process. Since a martingale measure automatically provides arbitrage free prices, we no longer have a problem of absence of arbitrage, but instead we have another problem. This is so because we now have the following two different formulas for bond prices

$$p(0,T) = \exp\left\{-\int_0^T f(0,s)ds\right\},$$

$$p(0,T) = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right],$$

where the short rate $r$ and the forward rates $f$ are connected by $r(t) = f(t, t)$. In order for these formulas to hold simultaneously, we have to impose some sort of consistency relation between $\alpha$ and $\sigma$ in the forward rate dynamics. The result is the famous Heath-Jarrow-Morton drift Condition.

**Proposition 18.2 (HJM drift Condition)** *Under the martingale measure $Q$, the processes $\alpha$ and $\sigma$ must satisfy the following relation, for every $t$ and every $T \geq t$.*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds. \tag{18.9}$$

**Proof** A short and brave argument is to observe that if we start by modelling directly under the martingale measure, then we may apply Proposition 18.1 with $\lambda = 0$. A more detailed argument is as follows.

From Proposition 15.5 we again have the bond price dynamics

$$dp(t,T) = p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\left\|S(t,T)\right\|^2\right\}dt + p(t,T)S(t,T)d\tilde{W}(t).$$

We also know that, under a martingale measure, the local rate of return has to equal the short rate $r$. Thus we have the equation

$$r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2 = r(t),$$

which gives us the result.

The moral of Proposition 18.2 is that when we specify the forward rate dynamics (under $Q$) we may freely specify the volatility structure. The drift parameters are then uniquely determined. An "algorithm" for the use of an HJM model can be written schematically as follows.

1. Specify, by your own choice, the volatilities $\sigma(t, T)$.
2. The drift parameters of the forward rates are now given by

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds. \tag{18.10}$$

3. Go to the market and observe today's forward rate structure

$$\{f^*(0,T);\, T \geq 0\}.$$

4. Integrate in order to get the forward rates as

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s). \tag{18.11}$$

5. Compute bond prices using the formula

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{18.12}$$

6. Use the results above in order to compute prices for derivatives.

To see at least how part of this machinery works we now study the simplest example conceivable, which occurs when the process $\sigma$ is a deterministic constant.

With a slight abuse of notation let us thus write $\sigma(t, T) \equiv \sigma$, where $\sigma > 0$. Equation (18.9) gives us the drift process as

$$\alpha(t,T) = \sigma \int_t^T \sigma ds = \sigma^2 (T - t),$$

so eqn (18.11) becomes

(18.13)

$$f(t,T) = f^{\star}(0,T) + \int_0^t \sigma^2 (T - s) ds + \int_0^t \sigma dW(s),$$

i.e.

(18.14)

$$f(t,T) = f^{*}(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W(t).$$

In particular we see that $r$ is given as

(18.15)

$$r(t) = f(t,t) = f^{*}(0,t) + \sigma^2 \frac{t^2}{2} + \sigma W(t),$$

so the short rate dynamics are

(18.16)

$$dr(t) = \left\{ f_T(0,t) + \sigma^2 t \right\} dt + \sigma dW(t),$$

(18.17)

which is exactly the Ho-Lee model, fitted to the initial term structure. Observe in particular the ease with which we obtained a perfect fit to the initial term structure.

## 18.3 The Musiela Parameterization

In many practical applications it is more natural to use time **to** maturity, rather than time **of** maturity, to parametrize bonds and forward rates. If we denote running time by $t$, time of maturity by $T$, and time to maturity by $x$, then we have $x = T - t$, and in terms of $x$ the forward rates are defined as follows.

**Definition 18.3** *For all* $x \geq 0$ *the forward rates* $r(t, x)$ *are defined by the relation*

$$r(t,x) = f(t, t + x).$$

(18.18)

Suppose now that we have the standard HJM-type model for the forward rates under a martingale measure $Q$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t).$$

(18.19)

The question is to find the $Q$-dynamics for $r(t, x)$, and we have the following result, known as the Musiela equation.

**Proposition 18.4 (The Musiela equation)** *Assume that the forward rate dynamics under Q are given by (18.19). Then*

$$dr(t,x) = \{ \mathcal{F} r(t,x) + D(t,x) \} dt + \sigma_0(t,x) dW(t), \tag{18.20}$$

*where*

$$\begin{aligned} \sigma_0(t,x) &= \sigma(t,t+x), \\ D(t,x) &= \sigma_0(t,x) \int_0^x \sigma_0(t,s)' ds, \\ \mathcal{F} &= \frac{\partial}{\partial x}. \end{aligned}$$

**Proof** Using a slight variation of the Itô formula we have

$$dr(t,x) = df(t,t+x) + \frac{\partial f}{\partial T}(t,t+x)dt,$$

where the differential in the term $df(t, t+x)$ only operates on the first $t$. We thus obtain

$$dr(t,x) = \alpha(t,t+x)dt + \sigma(t,t+x)dW(t) + \frac{\partial}{\partial x} r(t,x)dt,$$

and, using the HJM drift Condition, we obtain our result.

The point of the Musiela parameterization is that it highlights eqn (18.20) as an infinite dimensional SDE. It has become an indispensible tool of modern interest rate theory.

## 18.4 Exercises

**Exercise 18.1** Show that for the Hull-White model

$$dr = (\Theta(t) - ar)dt + \sigma dW$$

the corresponding HJM formulation is given by

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)} dW.$$

**Exercise 18.2** (**Gaussian interest rates** ) Take as given an HJM model (under the risk neutral measure $Q$) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where the volatility $\sigma(t, T)$ is a **deterministic** function of $t$ and $T$.

(a) Show that all forward rates, as well as the short rate, are normally distributed.
(b) Show that bond prices are log-normally distributed.

**Exercise 18.3** Consider the domestic and a foreign bond market, with bond prices being denoted by $p_d(t, T)$ and $p_f(t, T)$ respectively. Take as given a standard HJM model for the domestic forward rates $f_d(t, T)$, of the form

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dW(t),$$

where $W$ is a multidimensional Wiener process under the **domestic** martingale measure $Q$. The foreign forward rates are denoted by $f_f(t, T)$, and their dynamics, still under the domestic martingale measure $Q$, are assumed to be given by

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW(t).$$

Note that the same vector Wiener process is driving both the domestic and the foreign bond market. The exchange rate $X$ (denoted in units of domestic currency per unit of foreign currency) has the $Q$ dynamics

$$dX(t) = \mu(t)X(t)dt + X(t)\sigma_X(t)dW(t).$$

Under a foreign martingale measure, the coefficient processes for the foreign forward rates will of course satisfy a standard HJM drift Condition, but here we have given the dynamics of $f_f$ under the domestic martingale measure $Q$. Show that under this measure the foreign forward rates satisfy the modified drift Condition

$$\alpha_f(t,T) = \sigma_f(t,T)\left\{\int_t^T \sigma_f{}'(t,s)ds - \sigma_X{}'(t)\right\}.$$

**Exercise 18.4** With notation as in the exercise above, we define the **yield spread** $g(t, T)$ by

$$g(t,T) = f_f(t,T) - f_d(t,T).$$

Assume that you are given the dynamics for the exchange rate and the domestic forward rates as above. You are also given the spread dynamics (again under the domestic measure $Q$) as

$$dg(t,T) = \alpha_g(t,T)dt + \sigma_g(t,T)dW(t).$$

Derive the appropriate drift Condition for the coefficient process $\alpha_g$ in terms of $\sigma_g$, $\sigma_d$ and $\sigma_X$ (but not involving $\sigma_f$).

**Exercise 18.5 A consol bond** is a bond which forever pays a constant continuous coupon. We normalize the coupon to unity, so over every interval with lenght $dt$ the consol pays $1 \cdot dt$. No face value is ever paid. The price $C(t)$, at time

$t$, of the consol is the value of this infinite stream of income, and it is obviously (why?) given by

$$C(t) = \int_t^\infty p(t,s)ds.$$

Now assume that bond price dynamics under a martingale measure $Q$ are given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW(t),$$

where $W$ is a vector valued $Q$-Wiener process. Use the heuristic arguments given in the derivation of the HJM drift Condition (see Section 15.2.2) in order to show that the consol dynamics are of the form

$$dC(t) = C(t)(r(t) - 1)dt + \sigma_C(t)dW(t),$$

where

$$\sigma_C(t) = \int_t^\infty p(t,s)v(t,s)ds.$$

## 18.5 Notes

The basic paper for this chapter is Heath *et al.* (1987). The Musiela parameterization was first systematically investigated in Musiela (1993), and developed further in Brace and Musiela (1994). In Shirakawa (1991), Björk (1995), Björk *et al.* (1997a,b), and Jarrow and Madan (1995) the theory has been extended to more general driving noise processes. There is a growing literature on defaultable bonds. See Merton (1974), Duffie and Singleton (1994), Leland (1994), Jarrow *et al* (1997) and Lando (1997). A number of authors have recently considered forward rate models with so called lognormal volatility structure. This approach leads to a theoretical justification of Black-type bond option formulas. See Miltersen *et al* (1997) and Brace *et al* (1997). Concerning practical estimation of the yield curve see Anderson *et al* (1996).

# 19 Change of Numeraire

## 19.1 Introduction

Earlier in this text we encountered two probability measures: the objective probability measure *P*, and the "risk neutral" martingale measure *Q*. In the present chapter we will introduce a whole new class of probability measures, including *Q* as a member of that class. These probability measures are connected to a technique called "change of numeraire", and in many practical situations the amount of computational work needed in order to obtain a pricing formula can be drastically reduced by a suitable choice of numeraire.

To get some feeling for where we are heading, let us consider the pricing problem for a contingent claim $\chi$, in a model with a stochastic short rate of interest $r(t)$. From the general theory we know that the price at $t = 0$ of $\chi$ is given by the formula

$$\Pi(0;\chi) = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\cdot\chi\right]. \tag{19.1}$$

The problem with this formula from a computational point of view is that in order to compute the expected value we have to get hold of the joint distribution (under *Q*) of the two stochastic variables $\int_0^T r(s)ds$ and $\chi$, and finally we have to integrate with respect to that distribution. Thus we have to compute a double integral, and in most cases this turns out to be rather hard work.

Let us now make the (extremely unrealistic) assumption that *r* and $\chi$ are independent under *Q*. Then the expectation above splits, and we have the formula

$$\Pi(0;\chi) = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right]\cdot E^Q[\chi],$$

which we may write as

$$\Pi(0;\chi) = p(0,T)\cdot E^Q[\chi]. \tag{19.2}$$

We now note that (19.2) is a **much** nicer formula than (19.1), since

- We only have to compute the single integral $E^Q[\chi]$ instead of the double integral $E^Q[\exp\{-\int_0^T r(s)ds\}\cdot\chi]$.
- The bond price $p(0, T)$ in formula (19.2) does not have to be computed theoretically at all. We can **observe** it (at $t = 0$) directly on the bond market.

The drawback with the argument above is that, in most concrete cases, *r* and χ are **not** independent under *Q*, and if χ is a contingent claim on an underlying bond, this is of course obvious. What may be less obvious is that even if χ is a claim on an underlying stock which is *P*-independent of *r*, it will still be the case that χ and *r* will be dependent (generically) under *Q*. The reason is that under *Q* the stock will have *r* as its local rate of return, thus introducing a *Q*-dependence.

This is the bad news. The good news is that there exists a **general** pricing formula (see Theorem 19.8), a special case of which reads as

$$\Pi(0;\chi) = p(0,T)\cdot E^T[\chi]. \tag{19.3}$$

Here $E^T$ denotes expectation w.r.t. the so called **forward neutral** measure $Q^T$, which we will discuss below. We see from this formula that we do indeed have the multiplicative structure of (19.2), but the price we have to pay for generality is that the measure $Q^T$ depends upon the choice of maturity date *T*.

In order to understand why formulas of this type have anything to do with the choice of numeraire let us give a very brief and informal argument.

We start by recalling that the risk neutral martingale measure *Q* has the property that for every choice of a price process $\Pi(t)$ for a traded asset, the quotient

$$\frac{\Pi(t)}{B(t)}$$

is a *Q*-martingale. The point here is that we have divided the asset price $\Pi(t)$ by the **numeraire** asset price $B(t)$. It is now natural to investigate whether this martingale property can be generalized to other choices of numeraires, and we are (optimistically) led to the following conjecture.

**Conjecture 19.1.1** *Consider a fixed financial market, and a fixed "numeraire" asset price process* $S_0(t)$ *on the market. Then there exists a probability measure, denoted* $Q^0$, *such that*

$$\frac{\Pi(t)}{S_0(t)}$$

*is a* $Q^0$*-martingale for every asset price process* $\Pi(t)$.

Let us for the moment assume that the conjecture is true. We then fix a certain date of maturity *T*, and we choose the bond price process $p(t,T)$ (for this fixed *T*) as numeraire. According to the conjecture there should then exist a probability measure, which we denote by $Q^T$, such that the quotient

$$\frac{\Pi(t)}{p(t,T)}$$

is a martingale under the measure $Q^T$, for every $\Pi(t)$ which is the price process of a traded asset. In particular we have (using the relation $p(T,T) = 1$)

$$\frac{\Pi(0)}{p(0,T)} = E^T\left[\frac{\Pi(T)}{p(T,T)}\right] = E^T[\Pi(T)], \tag{19.4}$$

where $E^T$ denotes expectation under $Q^T$. Let us now choose the derivative price process $\Pi(t;\chi)$ as $\Pi(t)$ above. Then we have $\Pi(T) = \Pi(T;\chi) = \chi$, and (19.4) becomes

$$\frac{\Pi(0;\chi)}{p(0,T)} = E^T[\chi], \tag{19.5}$$

which gives us the formula (19.3).

The main object of this chapter is to prove the truth of the conjecture above. This is done in Theorem 19.8, and the rest of the chapter is devoted to applications of this technique.

# 19.2 The Normalized Economy

As our exogenously given model we take a factor model of the type discussed in Section 10.3, with the difference that we allow for stochastic rate of interest. Thus, by $X = (X^1, \ldots, X^{k+1})$ we will denote the factors of the model, with the notational convention that $X^{k+1}$ is always the short rate $r$. We make no a priori market assumptions, so whether or not a particular component is the price process of a traded asset in the market will depend on the particular application. It turns out that it is convenient to specify the model directly under a fixed martingale measure $Q$.

**Assumption 19.2.1** *The following objects are given a priori.*

- *An empirically observable $(k + 1)$-dimensional stochastic process*

$$X = (X^1, \ldots, X^{k+1}),$$

  *with the notational convention*

$$X^{k+1}(t) = r(t).$$

- *A fixed martingale measure $Q$, under which the factor dynamics have the form*

$$dX^i(t) = \mu^i(t,X(t))dt + \delta^i(t,X(t))dW(t),\ i = 1, \ldots, k+1,$$

  *where $W = (W_1, \ldots, W_d)^*$ is a standard d-dimensional $Q$-Wiener process.*
- *A risk free asset (money account) with the dynamics*

$$dB(t) = r(t)B(t)dt. \tag{19.6}$$

From Proposition 10.9 we recall the following fact.

**Proposition 19.1** *The price process for a given simple claim* $Y = \Phi(X(T))$ *is given by* $\Pi(t, Y) = F(t,X(t))$, *where $F$ is defined by*

$$F(t,x) = E^Q_{t,x}\left[\exp\left\{-\int_t^T r(s)ds\right\} \cdot Y\right]$$

For the model above we will now consider a fixed price vector $S$, consisting of $n$ arbitrage free asset price processes $S_0(t), \ldots, S_{n-1}(t)$. We could in fact consider the infinite set of **all** conceivable price processes in our model, but in order not to get into conceptual problems concerning an infinite set of assets we confine ourselves to a finite set. This set can, however, be chosen completely arbitrarily, subject only to the condition that all price processes can be interpreted as prices for **simple** contingent claims. It turns out to be convenient to adjoin $B$ as the $n$th component of the price vector above, and we now make the formalized assumption, while also setting the notation.

**Assumption 19.2.2**

- *We consider a fixed set of price processes $S_0(t), \ldots, S_n(t)$, each of which is assumed to be the price process for some simple contingent claim in the model above.*
- *Under the martingale measure $Q$, the $S$-dynamics are written as*

$$dS_i(t) = r(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}(t, X(t))dW_j(t), \tag{19.7}$$

  *for $i = 0, \ldots, n-1$.*
- *The nth asset price is always given by*

$$S_n(t) = B(t),$$

  *and thus (19.7) also holds for $i = n$ with $\sigma_{nj} = 0$ for $j = 1, \ldots, d$.*

**Example 19.2** A typical example of the setup above is the following extension of the Black–Scholes model (under $Q$).

$$dS_0(t) = r(t)S_0(t)dt + \sigma S_0(t)dW_1(t), \tag{19.8}$$

$$dr(t) = \mu(t, r(t))dt + \delta(t, r(t))dW_2(t), \tag{19.9}$$

$$dB(t) = r(t)B(t)dt. \tag{19.10}$$

In terms of our notation the factor vector $X$ is given by

$$X = (X^1, X^2) = (S, r),$$

i.e. one of the factors ($X^1 = S_0$) is a price process, whereas the other ($X^2 = r$) is not. In this setting it would be natural to study (among other things) the price vector $S = (S_0, S_1)$ with $S_0$ as above and $S_1$ a bond price process:

$$S_1(t) = p(t, T_1)$$

where $T_1$ is some fixed date of maturity.

**Remark 19.2.1** The assumption that all prices above are prices for **simple** contingent claims is needed for technical reasons, to ensure that the volatility components $\sigma_{ij}$ above are functions of the state vector $X$ at time $t$, rather than adapted functionals of the entire state trajectory over the interval $[0, t]$ (which would take us outside the scope of this book).

We now go on to introduce the numeraire process.

**Assumption 19.2.3** *We assume the existence of a stochastic process* $\beta$, *referred to as the* **numeraire** *process, which is assumed to satisfy (under Q) an SDE of the form*

$$d\beta(t) = \mu_\beta(t, X(t))dt + \sigma_\beta(t, X(t))dW(t). \tag{19.11}$$

*The numeraire is furthermore assumed to have the property that* $\beta(t) > 0$, *with probability* 1, *for each* $t \geq 0$.

**Remark 19.2.2** Note that, at this stage, the numeraire process could be almost any positive process. It can be a financial index, like a price index or the price of a traded asset, but it can also be a nonfinancial index, like the temperature (measured in Kelvin) at some given spot on the globe. The single most important special case occurs when $\beta$ is one of the asset prices $S_0, \ldots, S_n$ above, so in that case $\beta$ will in fact be the price of a traded asset. There are however interesting applications when $\beta$ is not the price of a traded asset, an obvious one being the case when $\beta$ equals a price index.

Now we go on to introduce the normalized economy, which simply means that instead of looking at the price vector process $S(t)$ we look at the normalized price vector process $S(t)/\beta(t)$.

**Definition 19.3** *The* **normalized economy** *(also referred to as the "Z-economy") is defined by the price vector process* $Z$, *where*

$$Z(t) = \frac{S(t)}{\beta(t)},$$

*i.e.*

$$Z(t) = [Z_0(t), \ldots, Z_n(t)] = \left[\frac{S_0(t)}{\beta(t)}, \frac{S_1(t)}{\beta(t)}, \frac{S_2(t)}{\beta(t)}, \ldots, \frac{S_n(t)}{\beta(t)}\right].$$

Our main objective is to price contingent claims of the form

$$\mathcal{Y} = \Phi(S(T), X(T)),$$

and the program to be carried out is schematically the following.

- Instead of pricing the claim $\mathcal{Y}$ directly in the $S$-economy, we study the normalized claim $\mathcal{Y}_Z = \mathcal{Y}/\beta(T)$.
- View the $Z$-model as a standard model for a financial market, and apply standard procedures in order to obtain the arbitrage free price (in $Z$-terms) for the claim $\mathcal{Y}_Z$.

- Transform the results back to the *S*-economy.

In order to implement the scheme above we have to deal with the problem of exactly how the *Z*-model is related to the *S*-model. In particular we have to investigate if and how arbitrage free prices in the *Z*-model are related to arbitrage free prices in the *S*-model. To be more precise, assume that a certain contract $\mathcal{Y}$ can be replicated in the *Z*-model, using a portfolio strategy *h*. We then have the following nontrivial questions.

- Is $\mathcal{Y}$ also replicable in the *S*-economy?
- If that is the case, how do we find the *S*-replicating portfolio?
- A given portfolio process *h* can, in our setting, be interpreted in two different ways: either as a portfolio in the *S*-model, or as a portfolio in the *Z*-model. Assume that *h* is self-financing in the *S*-model. Is it then the case that *h* is also self-financing in the *Z*-model?

In order to analyze the problems we need some notation.

**Definition 19.4**

- *A* **portfolio strategy** *is any adapted* $(n+1)$*-dimensional process*

$$h(t) = [h_0(t), h_1(t), \dots, h_n(t)]$$

- *The* **S-value process** $V^S(t;h)$ *corresponding to the portfolio h is given by*

$$V^S(t,h) = \sum_{i=0}^{n} h_i(t)S_i(t). \tag{19.12}$$

- *The* **Z-value process** $V^Z(t;h)$ *corresponding to the portfolio h is given by*

$$V^Z(t,h) = \sum_{i=0}^{n} h_i(t)Z_i(t). \tag{19.13}$$

- *A portfolio is said to be* **S-self-financing** *if*

$$dV^S(t,h) = \sum_{i=0}^{n} h_i(t)dS_i(t). \tag{19.14}$$

- *A portfolio is said to be* **Z-self-financing** *if*

$$dV^Z(t,h) = \sum_{i=0}^{n} h_i(t)dZ_i(t). \tag{19.15}$$

- *A contingent T-claim* $\mathcal{Y}$ *is said to be* **S-reachable** *if there exists an S-self-financing portfolio strategy h such that*

$$V^S(T;h) = \mathcal{Y},$$

*with probability* 1.

- *A contingent T-claim* Y *is said to be* **Z-reachable** *if there exists a Z-self-financing portfolio strategy h such that*

$$V^Z(T; h) = \mathrm{Y},$$

*with probability* 1.

The intuitive feeling is that the concept of a self-financing portfolio should not depend upon the particular choice of numeraire. That this is indeed the case is shown by the main result of this section, which is the invariance lemma below. It says roughly that the *S*-model and the *Z*-model are equivalent in a very strong sense.

**Lemma 19.5 (Invariance lemma)** *With assumptions and notation as above, the following hold.*

*(i) The value processes $V^S$ and $V^Z$ are connected by*

$$V^Z(t, h) = \frac{1}{S_0(t)} \cdot V^S(t, h).$$

*(ii) A portfolio h is S-self-financing if and only if it is Z-self-financing.*
*(iii) A claim* Y *is S-reachable if and only if the claim*

$$\frac{\mathrm{Y}}{S_0(T)}$$

*is Z-reachable.*

**Proof** The relation (i) follows immediately from the definitions, and (iii) follows from (i) and (ii), so it only remains to prove (ii). Assume therefore that the portfolio *h* is *S*-self-financing. Denoting the scalar product between vectors by the "scalar dot" ·, using the notation $\alpha(t) = \beta^{-1}(t)$, and suppressing the *t*-variable, we have from this assumption that

$$Z = \beta^{-1}S,$$

$$V^S = h \cdot S, \tag{19.16}$$

$$V^Z = \beta^{-1}V^S, \tag{19.17}$$

$$dV^S = h \cdot dS. \tag{19.18}$$

(19.19)

We now want to prove that in fact

$$dV^Z = h \cdot dZ.$$

Using the Itô formula on $Z = \beta^{-1}S$, we thus want to prove that

$$dV^Z = \beta^{-1}h \cdot dS + h \cdot Sd\beta^{-1} + h \cdot dSd\beta^{-1}.$$

(19.20)

Now, from (19.18) we have

$$dV^Z = \beta^{-1}dV^S + V^S d\beta^{-1} + d\beta^{-1}dV^S.$$

Substituting (19.17) and (19.19) into this equation gives

$$dV^Z = \beta^{-1}h \cdot dS + h \cdot S d\beta^{-1} + d\beta^{-1}h \cdot S,$$

which is what we wanted to prove.

In the spirit of Definition 19.4, we may define the concepts of $S$-arbitrage portfolios and $Z$-arbitrage portfolios, $S$-completeness and $Z$-completeness, and so on, but these definitions should by now be obvious. We have the following easy result.

**Proposition 19.6** *With notation as above, the following hold.*

- *The S-market is arbitrage free if and only if the Z-market is arbitrage free.*
- *The S-market is complete if and only if the Z-market is complete.*
- *The process* $\Pi$ *(t) is an arbitrage free price process in the S-model if and only if the process*

$$\frac{\Pi(t)}{\beta(t)}$$

*is an arbitrage free price process in the Z-model.*

**Proof** Since both arbitrage portfolios and reachable claims are defined entirely in terms of self-financing portfolios, the results follow directly from the invariance lemma.

## 19.3 Pricing

We take as given the model of the previous section, as well as a $T$-claim $\mathrm{Y}$. Observe now that the claim □ can be considered as a claim, either in the $S$-market, or in the $Z$-market. The corresponding arbitrage free prices of the claim will be denoted by $\Pi(t, \mathrm{Y})$ and $\Pi^Z(t, \mathrm{Y})$ respectively. We then have the following simple but useful result.

**Lemma 19.7** The following relation holds:

$$\Pi(t, \mathrm{Y}) = \beta(t) \cdot \Pi^Z(t, \frac{\mathrm{Y}}{\beta(T)}).$$

**Proof** We know by definition that $\Pi(t, \mathrm{Y})$ is an arbitrage free price process in the $S$-market, and from Proposition 19.6 it then follows that

$$\frac{\Pi(t, \mathrm{Y})}{\beta(t)}$$

is an arbitrage free price process in the $Z$-market. It only remains to compute the value of this latter price process at time $T$, and using the relation $\Pi(T; \mathrm{Y}) = \mathrm{Y}$ we obtain

$$\frac{\Pi(T; \mathrm{Y})}{\beta(T)} = \frac{\mathrm{Y}}{\beta(T)},$$

which gives us the result.

Thus we see that the problem of computing the $S$-price of $\mathcal{Y}$ is reduced to the problem of computing the $Z$-price of $\frac{\mathcal{Y}}{\beta(T)}$. At this level of generality, very little can be said about this price, but if we assume that the numeraire process is the price process of a traded asset, we can obtain much stronger results.

**Assumption 19.3.1** *For the rest of this section, the numeraire process* $\beta$ *is assumed to be the* **price process of a traded asset** . *In technical terms we thus assume that* $\beta$ *is the arbitrage free price process for some (simple) contingent claim, and without loss of generality we may assume that this asset has index number zero, i.e.* $\beta(t) = S_0(t)$.

**Remark 19.3.1** Note that $S_0$ does not have to be one of the "underlying" assets in the a priori given model. It can equally well be a derivative of the underlying, in which case we simply adjoin it to the original model, and give it index number zero.

Because of this assumption and our normalization, we have a very simple structure for $Z$-prices, and in order to see this, we recall that in the $Z$-market we have one particularly simple asset price, namely the process

$$Z_0(t) = \frac{S_0(t)}{S_0(t)} \equiv 1.$$

Thus the $Z$-market is a market where **the riskless rate of return equals zero** , and this fact immediately gives us the following fundamental result, which shows the correctness of Conjecture 19.11.

**Theorem 19.8 (Main theorem)** *Let the numeraire* $S_0$ *be the price process for a traded assset with* $S_0(t) > 0$ *for all t. Then there exists a probability measure, denoted by* $Q^0$, *with the following properties.*

- *For every T-claim* $\mathcal{Y}$ , *we have*

$$\Pi(t, \mathcal{Y}) = S_0(t) E^0_{t,X(t)} \left[ \frac{\mathcal{Y}}{S_0(T)} \right], \tag{19.21}$$

*where* $E^0$ *denotes expectation w.r.t.* $Q^0$.

- *The* $Q^0$*-dynamics of the Z-processes are given by*

$$dZ_i = Z_i [\sigma_i - \sigma_0] dW^0, \quad i = 0, \ldots, n. \tag{19.22}$$

- *The* $Q^0$*-dynamics of the price processes are given by*

$$dS_i = S_i (r + \sigma_i \sigma_0^*) dt + S_i \sigma_i dW^0, \tag{19.23}$$

*where* $W^0$ *is a* $Q^0$*-Wiener process.*

- *The* $Q^0$*-dynamics of the X-processes are given by*

$$dX^i(t) = \{\mu^i + \delta^i \sigma_0^*\} dt + \delta^i dW^0(t). \tag{19.24}$$

- *The measure $Q^0$ depends upon the choice of numeraire asset $S_0$, but the same measure is used for all claims, regardless of their exercise dates.*

**Proof** Since, in the $Z$-model, the riskless rate equals zero, it follows from Proposition 10.6 that there exists a measure $Q^0$, such that

$$\Pi^Z(t, \xi) = E^0_{t,X(t)}[\xi]$$

for any contingent $T$-claim $\xi$. This formula (with $\xi = \mathcal{Y} / S_0(T)$) together with Lemma 19.7 gives us the pricing formula (19.21). The $Q$-dynamics of $Z_i$ are easily obtained as

$$dZ_i = Z_i(\|\sigma_0\|^2 - \sigma_i\sigma_0^*)dt + Z_i(\sigma_i - \sigma_0)dW$$

and from the fact that, under a martingale measure, all price processes have a local rate of return equal to the short rate of interest, while the diffusion part remains unchanged, we have (19.22) for the $Q^0$-dynamics for $Z_i$.

Since by definition $Z_n = \frac{B(t)}{S_0(t)}$ we can obtain the $Q^0$-dynamics of $S_0$ by writing $S_0 = B/Z_n$, applying Itô's formula on the quotient, and using the $Q^0$-dynamics for $Z_n$ given by (19.22). (The $B$-dynamics are the same under $Q^0$ as under $Q$.) We can then write $S_i = Z_i \cdot S_0$, and use the $S_0$-dynamics just obtained together with (19.22) in order to get the $Q^0$-dynamics for $S_i$ for $i \neq 0$. Thus we have identified the $\lambda$-process of Proposition 10.6 as $\lambda = -\sigma_0$, and now the factor dynamics (19.24) follow from Proposition 10.6.

**Remark 19.3.2** We emphasize again that the reason why we need the numeraire process to be the price of a traded asset in the theorem above is that this assumption implies that the normalized economy has a risk free asset with zero rate of return. In a more abstract approach the "tradability" assumption is formalized by an assumption that the numeraire asset has the property that $S_0(t)/B(t)$ is a martingale under the (not necessarily unique) risk neutral martingale measure $Q$.

**Remark 19.3.3** For the probabilist we note that the result above implies that the likelihood process

$$L(t) = \left.\frac{dQ^0}{dQ}\right|_{\mathcal{F}_t}$$

has the dynamics

$$dL = L\sigma_0 dW,$$

i.e. the Girsanov transformation is determined by the numeraire volatility.

The main point of Theorem 19.8 is the pricing formula (19.21), which is particularly useful when the claim $\mathcal{Y}$ is of the form

$$\mathcal{Y} = S_0(T) \cdot \xi,$$

since then we obtain the simple expression

$$\Pi(t, \mathcal{Y}) = S_0(t)E^0_{t,X(t)}[\xi].$$

A typical example when this situation occurs is when we want to compute derivatives of several underlying assets. Assume for example that we are given two asset prices $S_0$ and $S_1$, as well as a stochastic rate of interest $r$, and that the contract $\mathcal{Y}$ to be priced is of the form $\mathcal{Y} = \Phi(S_0(T), S_1(T))$, where $\Phi$ is a given **linearly homogeneous** (see Section 9.4) function. Using the standard machinery we would have to compute the price as

$$\Pi(t, X) = E^Q_{t,X(t)}\left[\exp\left\{-\int_t^T r(s)ds\right\}\Phi(S_0(T), S_1(T))\right],$$

which essentially amounts to the calculation of a triple integral. If we instead use $S_0$ as numeraire we have

$$\Pi(t, X) = S_0(t)E^0_{t,X(t)}[\varphi(Z_1(T))]\ \mathcal{F}_t, \tag{19.25}$$

where $\varphi(z) = \Phi(1, z)$ and $Z_1(T) = S_1(T)/S_0(T)$. Note that the factor $S_0(t)$ is the price of the traded asset $S_0$ at time $t$, so this quantity does not have to be computed—it can be directly observed on the market. Thus the computational work is reduced to that of computing a single integral, and for the actual computation we use the $Q^0$-dynamics of $Z$ given by (19.22).

**Example 19.9** We now go back to Example 9.5 to see how the results above can be applied. Assume that we have two stocks, $S_0$ and $S_1$, with price processes of the following form under the objective probability $P$.

$$dS_0 = \alpha_0 S_0 dt + \sigma_0 S_0 d\tilde{W}_0, \tag{19.26}$$

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 d\tilde{W}_1. \tag{19.27}$$

Here $\bar{W}_0$ and $\bar{W}_1$ are assumed to be independent $P$-Wiener processes, but it would also be easy to treat the case when there is a coupling between the two assets.

Under the standard risk neutral measure $Q$ (with $B$ as numeraire) the price dynamics will be given by

$$dS_0 = rS_0 dt + \sigma_0 S_0 dW_0, \tag{19.28}$$

$$dS_1 = rS_1 dt + \sigma_1 S_1 dW_1. \tag{19.29}$$

The $T$-claim to be priced is an **exchange option** , which gives the holder the right, but not the obligation, to exchange one $S_0$ share for one $S_1$ share at time $T$. Formally this means that the claim is given by $\mathcal{Y} = \max[S_1(T) - S_0(T), 0]$, and we note that we have a linearly homogeneous contract function. From Theorem 19.8, and using homogeneity, the price at $t = 0$ is given by

$$\Pi(0; Y) = S_0(0)E^0[\max[Z_1(T) - 1, 0]],$$

with $Z_1(t) = S_1(t)/S_0(t)$. We are thus in fact valuing a European call option on $Z_1(T)$, with strike price $K = 1$. Again from Theorem 19.8 we have the $Z$-dynamics under $Q^0$ as

$$dZ_1 = Z_1[\sigma_1 dW_1^0 - \sigma_0 dW_0^0],$$

which we can write as

$$dZ_1 = Z_1\sqrt{\sigma_0^2 + \sigma_1^2}\,dV^0$$

where $V^0$ is a $Q^0$-Wiener process. The price is thus given by the formula

$$\Pi(0; X) = S_0(0) \cdot c(0, Z_1(0)).$$

Here $c(t, z_1)$ is given directly by the Black–Scholes formula as the price of a European call option, valued at 0, with time of maturity $T$, strike price $K = 1$, short rate of interest $r = 0$, on a stock with volatility $\sqrt{\sigma_0^2 + \sigma_1^2}$ and price $z_1$.

# 19.4 Forward Measures

## 19.4.1 Using the T-Bond as Numeraire

We now go on to apply the general results above to the bond market. Let us therefore consider an arbitrary $T$-claim $Y$, and choose the numeraire $S_0$ as the bond maturing at $T$, i.e. $S_0(t) = p(t, T)$. Note that the maturity date of the bond coincides with the maturity date of the derivative. From Theorem 19.8 we infer the existence of a measure $Q^T$, known as the **T-forward neutral** measure. such that

$$\Pi(t, Y) = p(t,T)E^T_{t,X(t)}\left[\frac{Y}{p(T,T)}\right], \tag{19.30}$$

which reduces to the following fundamental formula.

**Proposition 19.10** *With $Q^T$ denoting the forward neutral measure, and $E^T$ denoting expectation under $Q^T$, the following relation holds for every T-claim $Y$.*

$$\Pi(t, Y) = p(t,T)E^T_{t,X(t)}[Y]. \tag{19.31}$$

We have thus finally proved the formula (19.3). Note that in the degenerate case when $r$ is deterministic, the forward measure $Q^T$ is equal to the risk neutral measure $Q$.

**Lemma 19.11** *If the short rate r is deterministic, then*

$$Q^T = Q,$$

*for every choice of maturity T.*

**Proof** In the case of a deterministic short rate, it is trivial to see that bond prices are given by

$$p(t,T) = \exp\left\{-\int_t^T r(s)ds\right\},$$

so $p(t,T) = B(t) \cdot \exp\{-\int_0^T r(s)ds\}$. Thus (19.30) is satisfied by setting $Q^T$ equal to $Q$.

The forward neutral measures are natural to use when dealing with interest rate derivatives, but for options (on bonds or stocks) there are even further simplifications to be made, which we will meet in the next section.

We finish this section with a note on the connection between $Q$ and $Q^T$. First we notice that for any contingent $T$-claim we have the relations

$$\Pi(0; Y) = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} Y\right]$$

$$\Pi(0; Y) = p(0,T)E^T[Y].$$

Generalizing this argument from the case $t = 0$ to an arbitrary $t$, we have the following results, which connect the forward neutral measures to the risk neutral measure.

**Proposition 19.12** *For any simple T-claim* $Y$, *the following relation holds.*

$$E^T[Y] = E^Q\left[\frac{\exp\{-\int_0^T r(s)ds\}}{p(0,T)} \cdot Y\right], \tag{19.32}$$

$$E^T_{t,X_t}[Y] = \frac{E^Q_{t,X_t}\left[\exp\left\{-\int_0^T r(s)ds\right\} \cdot Y\right]}{p(t,T)}. \tag{19.33}$$

**Remark 19.4.1** The probabilist will identify

$$\frac{\exp\{-\int_0^T r(s)ds\}}{p(0,T)}$$

as the Radon–Nikodym derivative $\frac{dQ^T}{dQ}$ (on $\mathcal{F}_T$).

## 19.4.2 An Expectation Hypothesis

We now move to the forward rate process $f(t, T)$. The economic interpretation of $f(t, T)$ is that this is the riskless rate of return which we may have on an investment over the infinitesimal interval $[T, T + dT]$ if the contract is made at $t$. On the other hand, the short rate $r(T)$ is the riskless rate of return over the infinitesimal interval $[T, T + dT]$, if the contract is made at $T$. Thus it is natural to view $f(t, T)$ (which can be observed at $t$) as an estimate of the future short rate $r(T)$. More explicitly it is argued that if the market expects the short rate at $T$ to be high, then it seems reasonable to assume that the forward rate $f(t, T)$ is also high, since otherwise there would be profits to be made on the bond market.

Our task now is to determine whether this reasoning is correct in a more precise sense, and to this end we study the most formalized version of the argument above, known as the "unbiased expectation hypothesis" for forward rates. If we set $t = 0$, this hypothesis says that in an efficient market we must have

$$f(0,T) = E[r(T)], \tag{19.34}$$

i.e. the forward rate is an unbiased estimator of the future spot rate. If we interpet the expression "an efficient market" as "an arbitrage free market" then we may use our general machinery to analyze the problem.

First we notice that there is no probability measure indicated in (19.34), so we have to make a choice.

Of course there is no reason at all to expect the hypothesis to be true under the objective measure $P$, but it is often claimed that it holds "in a risk neutral world". This more refined version of the hypothesis can then be formulated as

$$f(0,T) = E^{Q}[r(T)], \tag{19.35}$$

where $Q$ is the usual risk neutral martingale measure.

We now use the definition

$$f(0,T) = -\frac{\partial}{\partial T}\ln p(0,T),$$

to obtain

$$f(0,T) = -\frac{\frac{\partial p(0,T)}{\partial T}}{p(0,T)}.$$

On the other hand we have

$$p(0,T) = E^{Q}\left[\exp\left\{-\int_0^T r(s)ds\right\}\right],$$

so, by differentiating under the expectation sign, we obtain

$$\frac{\partial p(0,T)}{\partial T} = -E^{Q}\left[r(T)\cdot\exp\left\{-\int_0^T r(s)ds\right\}\right].$$

We thus have

$$f(0,T) = E^{Q}\left[r(T)\cdot\frac{\exp\{-\int_0^T r(s)ds\}}{p(0,T)}\right], \tag{19.36}$$

which shows that the conjecture (19.35) is generically **not** true, except of course for the case when $r$ is deterministic.

For a more positive result, we see from formula (19.32) that in fact we have the following useful relation.

**Lemma 19.13**

$$f(0,T) = E^T[r(T)]. \tag{19.37}$$

Thus the expectation hypothesis turns out to be true under the forward neutral measure. Note, however, that we have different measures for different maturities *T*.

# 19.5 A General Option Pricing Formula

The object of this section is to give a fairly general formula for the pricing of European call options. Assume therefore that we are given a financial market with a (possibly stochastic) short rate of interest *r*, and a strictly positive asset price process $S(t)$.

The option under consideration is a European call on *S* with date of maturity *T* and strike price *K*. We are thus considering the *T*-claim

$$\chi = \max[S(T) - K, 0], \tag{19.38}$$

and, for readability reasons, we confine ourselves to computing the option price $\Pi(t;\chi)$ at time $t = 0$.

The main reason for the existence of a large number of explicit option pricing formulas is that the contract function for an option is piecewise linear. We can capitalize on this fact by using a not so well known trick with indicator functions. Write the option as

$$\chi = [S(T) - K] \cdot I\{S(T) \geq K\},$$

where *I* is the indicator function, i.e.

$$I\{S(T) \geq K\} = \begin{cases} 1 \text{ if } S(T) \geq K \\ 0 \text{ if } S(T) < K. \end{cases}$$

We obtain

$$\begin{aligned} \Pi(0;\chi) &= E^Q[B^{-1}(T)[S(T) - K]I\{S(T) \geq K\}] \\ &= E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} S(T) \cdot I\{S(T) \geq K\}\right] \\ &\quad - KE^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} \cdot I\{S(T) \geq K\}\right]. \end{aligned}$$

In the first term above, we use the measure $Q^S$ having *S* as numeraire, and for the second term we use the *T*-forward measure. From Theorem 19.8 and formula (19.31) we then obtain the following basic option pricing formula, which is a substantial extension of the standard Black–Scholes formula.

**Proposition 19.14 (General option pricing formula)** *Given the assumptions above, the option price is given by*

$$\Pi(0;\chi) = S(0)Q^S(S(T) \geq K) - Kp(0,T)Q^T(S(T) \geq K). \tag{19.39}$$

*Here $Q^T$ denotes the T-forward measure, whereas $Q^S$ denotes the martingale measure for the numeraire process $S(t)$.*

In order to use this formula in a real situation we have to be able to compute the probabilities above, and the standard Condition which ensures computability turns out to be that volatilities should be deterministic. Hence we have the following assumption.

**Assumption 19.5.1** *Assume that the process $Z_{S,T}$ defined by*

$$Z_{S,T}(t) = \frac{S(t)}{p(t,T)} \tag{19.40}$$

*has a stochastic differential of the form*

$$dZ_{S,T}(t) = Z_{S,T}(t)m_{S,T}(t)dt + Z_{S,T}(t)\sigma_{S,T}(t)dW, \tag{19.41}$$

*where the volatility process $\sigma_{S,T}(t)$ is* **deterministic** .

The crucial point here is of course the assumption that the row-vector process $\sigma_{S,T}$ is deterministic. Note that the volatility process as always is unaffected by a change of measure, so we do not have to specify under which measure we check the Condition. It can be done under $P$ as well as under $Q$.

We start the computations by writing the probability in the second term of (19.39) as

$$Q^T(S(T) \geq K) = Q^T\left(\frac{S(T)}{p(T,T)} \geq K\right) = Q^T(Z_{S,T}(T) \geq K). \tag{19.42}$$

Since $Z_{S,T}$ is an asset price, normalized by the price of a $T$-bond, it has zero drift (Theorem 19.8) under $Q^T$, so its $Q^T$-dynamics are given by

$$dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW^T. \tag{19.43}$$

This is basically GBM, driven by a multidimensional Wiener process, and it is easy to see that the solution is given by

$$Z_{S,T}(T) = \frac{S(0)}{p(0,T)}\exp\left\{-\frac{1}{2}\int_0^T \left\|\sigma_{S,T}\right\|^2(t)dt + \int_0^T \sigma_{S,T}(t)dW^T\right\}. \tag{19.44}$$

In the exponent we have a stochastic integral and a deterministic time integral. Since the integrand in the stochastic integral is deterministic, an easy extension of Lemma 3.15 shows that the stochastic integral has a Gaussian distribution with zero mean and variance

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt. \tag{19.45}$$

The entire exponent is thus normally distributed, and we can write the probability in the second term in (19.39) as

$$Q^T(S(T) \geq K) = N[d_2],$$

where

$$d_2 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}. \tag{19.46}$$

Since the first probability term in (19.39) is a $Q^s$-probability, it is natural to write the event under consideration in terms of a quotient with $S$ in the denominator. Thus we write

$$Q^S(S(T) \geq K) = Q^S\left(\frac{p(T,T)}{S(T)} \leq \frac{1}{K}\right) = Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right), \tag{19.47}$$

where $Y_{S,T}$ is defined by

$$Y_{S,T}(t) = \frac{p(t,T)}{S(t)} = \frac{1}{Z_{S,T}(t)}.$$

Again using Theorem 19.8 we see that under $Q^s$ the process $Y_{S,T}$ has zero drift, so its $Q^s$-dynamics are of the form

$$dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW^S.$$

Since $Y_{S,T} = Z^{-1}_{S,T}$, an easy application of Itô's formula gives us $\delta_{S,T}(t) = -\sigma_{S,T}(t)$. Thus we have

$$Y_{S,T}(T) = \frac{p(0,T)}{S(0)}\exp\left\{-\frac{1}{2}\int_0^T \left\|\sigma_{S,T}\right\|^2(t)dt - \int_0^T \sigma_{S,T}(t)dW^S\right\},$$

and again we have a normally distributed exponent. Thus, after some simplification,

$$Q^S(S(T) \geq K) = N[d_1],$$

where

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}. \tag{19.48}$$

We have thus proved the following result.

**Proposition 19.15 (Option pricing)** *Under the conditions given in Assumption 19.5.1, the price of the call option defined in (19.38) is given by the formula*

$$\Pi(0;\chi) = S(0)N[d_1] - K \cdot p(0,T)N[d_2]. \tag{19.49}$$

*Here* $d_2$ *and* $d_1$ *are given in (19.46) and* (19.48) *respectively, whereas* $\Sigma^2_{S,T}(T)$ *is defined by (19.45).*

# 19.6 The Hull-White Model

As a concrete application of the option pricing formula of the previous section, we will now consider the case of interest rate options in the Hull-White model (extended Vasiček). To this end recall that in the Hull-White model the $Q$-dynamics of $r$ are given by

$$dr = \{\Phi(t) - ar\}\,dt + \sigma dW. \tag{19.50}$$

From Section 17.3 we recall that we have an affine term structure

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}, \tag{19.51}$$

where $A$ and $B$ are deterministic functions, and where $B$ is given by

$$B(t,T) = \frac{1}{a}\{1 - e^{-a(T-t)}\}. \tag{19.52}$$

The project is to price a European call option with date of maturity $T_1$ and strike price $K$, on an underlying bond with date of maturity $T_2$, where $T_1 < T_2$. In the notation of the general theory above this means that $T = T_1$ and that $S(t) = p(t, T_2)$. We start by checking Assumption 19.5.1, i.e. if the volatility, $\sigma_z$, of the process

$$Z(t) = \frac{p(t,T_2)}{p(t,T_1)} \tag{19.53}$$

is deterministic. (In terms of the notation in Section 19.5$Z$ corresponds to $Z_{S,T}$ and $\sigma_z$ corresponds to $\sigma_{S,T}$.)

Inserting (19.51) into (19.53) gives

$$Z(t) = \exp\{A(t,T_2) - A(t,T_1) - [B(t,T_2) - B(t,T_1)]\,r(t)\}.$$

Applying the Itô formula to this expression, and using (19.50), we get the $Q$-dynamics

$$dZ(t) = Z(t)\{\dots\}\,dt + Z(t)\cdot\sigma_z(t)dW, \tag{19.54}$$

where

$$\sigma_z(t) = -\sigma[B(t,T_2) - B(t,T_1)] = \frac{\sigma}{a}e^{at}[e^{-aT_2} - e^{-aT_1}]. \tag{19.55}$$

Thus $\sigma_z$ is in fact deterministic, so we may apply Proposition 19.15. We obtain the following result, which also holds (why?) for the Vasiček model.

**Proposition 19.16 (Hull-White bond option)** *In the Hull-White model* (*19.50*) *the price, at* $t = 0$, *of a European call with strike price K, and time of maturity* $T_1$, *on a bond maturing at* $T_2$ *is given by the formula*

$$\Pi(0;\chi) = p(0,T_2)N[d_1] - K \cdot p(0,T_1)N[d_2],$$

*where*

(19.56)

$$d_2 = \frac{\ln\left(\frac{p(0,T_2)}{K_p(0,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}},$$

$$d_1 = d_2 + \sqrt{\Sigma^2}, \tag{19.57}$$

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\{1 - e^{-2aT_1}\}\left\{1 - e^{-a(T_2-T_1)}\right\}^2. \tag{19.58}$$

(19.59)

We end the discussion of the Hull-White model, by studying the pricing problem for a claim of the form

$$Z = \Phi(r(T)).$$

Using the $T$-bond as numeraire, we have, from Proposition 19.10 and with $r$ as the only $X$-factor,

$$\Pi(t;Z) = p(t,T)E^T_{t,r}[\Phi(r(T))],$$

(19.60)

so we must find the distribution of $r(T)$ under $Q^T$, and to this end we will use Theorem 19.8 (with $p(t, T)$ chosen as $S_0$). We thus need the volatility of the $T$-bond, and from (19.51)–(19.52) we obtain bond prices (under $Q$) as

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dW,$$

where the volatility $v(t, T)$ is given by

(19.61)

$$v(t,T) = -\sigma B(t,T).$$

(19.62)

Thus, using Theorem 19.8 the $Q^T$-dynamics of the short rate are given by

$$dr = [\Theta(t) - ar - \sigma^2 v(t,T)]dt + \sigma dW^T,$$

(19.63)

where $W^T$ is a $Q^T$-Wiener process.

We observe that, since $v(t, T)$ and $\Theta(t)$ are deterministic, $r$ is a Gaussian process, so the distribution of $r(T)$ is completely determined by its mean and variance under $Q^T$. Solving the linear SDE (19.63) gives us

$$\begin{aligned} r(T) &= e^{-a(T-t)} + \int_t^T e^{-a(T-s)}[\Theta(s) - \sigma^2 v(s,T)]ds \\ &+\sigma\int_t^T e^{-a(T-s)}dW^T(s). \end{aligned}$$

(19.64)

We can now compute the conditional $Q^T$-variance of $r(T)$, $\sigma_r^2(t,T)$, as

$$\sigma_r^2(t,T) = \sigma^2 \int_t^T e^{-2a(T-s)} ds = \frac{\sigma^2}{2a} \{1 - e^{-2a(T-t)}\}. \tag{19.65}$$

Note that the $Q^T$-mean of $r(T)$, $m_r(t,T) = E_{t,r}^T[r(T)]$, does not have to be computed at all. We obtain it directly from Lemma 19.13 as

$$m_r(t,T) = f(t,T),$$

which can be observed directly from market data.

Under $Q^T$, the conditional distribution of $r(T)$ is thus the normal distribution $N[f(t, T), \sigma_r(t, T)]$, and performing the integration in (19.60) we have the final result.

**Proposition 19.17** *Given the assumptions above, the price of the claim* $X = \Phi(r(T))$ *is given by*

$$\Pi(t, X) = p(t,T) \frac{1}{\sqrt{2\pi\sigma_r^2(t,T)}} \int_{-\infty}^{\infty} \Phi(z) \exp\left\{ -\frac{[z - f(t,T)]^2}{2\sigma_r^2(t,T)} \right\} dz, \tag{19.66}$$

*where* $\sigma_r^2(t,T)$ *is given by (19.65).*

# 19.7 The General Gaussian Model

In this section we extend our earlier results, by computing prices of bond options in a general Gaussian forward rate model. We specify the model (under $Q$) as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t), \tag{19.67}$$

where $W$ is a $d$-dimensional $Q$-Wiener process.

**Assumption 19.7.1** *We assume that the volatility vector function*

$$\sigma(t,T) = [\sigma_1(t,T), \ldots, \sigma_d(t,T)]$$

*is a* **deterministic** *function of the variables t and T.*

Using Proposition 15.5 the bond price dynamics under $Q$ are given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW(t), \tag{19.68}$$

where the volatility is given by

$$v(t,T) = -\int_t^T \sigma(t,s)ds. \tag{19.69}$$

We consider a European call option, with expiration date $T_0$ and exercise price $K$, on an underlying bond with maturity $T_1$ (where of course $T_0 < T_1$). In order to compute the price of the bond, we use Proposition 19.15, which means that we first have to find the volatility $\sigma_{T_1,T_0}$ of the process

$$Z(t) = \frac{p(t,T_1)}{p(t,T_0)}.$$

An easy calculation shows that in fact

$$\sigma_{T_1,T_0}(t) = v(t,T_1) - v(t,T_0) = -\int_{T_0}^{T_1} \sigma(t,s)ds. \tag{19.70}$$

This is clearly deterministic, so Assumption 19.5.1 is satisfied. We now have the following pricing formula.

**Proposition 19.18 (Option prices for Gaussian forward rates)** *The price, at $t = 0$, of the option*

$$\chi = \max\left[p(T_0,T_1) - K,0\right]$$

*is given by*

$$\Pi(0;\chi) = p(0,T_1)N[d_1] - K \cdot p(0,T_0)N[d_2], \tag{19.71}$$

*where*

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{p(0,T_1)}{Kp(0,T_0)}\right) + \frac{1}{2}\Sigma^2_{T_1,T_0}}{\sqrt{\Sigma^2_{T_1,T_0}}}, \\ d_2 &= d_1 - \sqrt{\Sigma^2_{T_1,T_0}}, \\ \Sigma^2_{T_1,T_0} &= \int_0^{T_0} \left\| \sigma_{T_1,T_0}(s) \right\|^2 ds, \end{aligned}$$

*and $\sigma_{T_1,T_0}$ is given by (19.70).*

**Proof** Follows immediately from Proposition 19.15.

# 19.8 Caps and Floors

The object of this section is to present one of the most traded of all interest rate derivatives—the cap—and to show how it can be priced.

An interest rate **cap** is a financial insurance contract which protects you from having to pay more than a prespecified rate, the **cap rate** , even though you have a loan at a floating rate of interest. There are also **floor** contracts which guarantee that the interest paid on a floating rate loan will never be below some predetermined **floor rate** . For simplicity we assume that we are standing at time $t = 0$, and that the cap is to be in force over the interval $[0,T]$. Technically speaking, a cap is the sum of a number of basic contracts, known as **caplets** , which are defined as follows.

- The interval $[0, T]$ is subdivided by the equidistant points

$$0 = T_0, T_1, \ldots, T_n = T.$$

We use the notation $\delta$ for the length of an elementary interval, i.e. $\delta = T_i - T_{i-1}$. Typically, $\delta$ is a quarter of a year, or half a year.
- The cap is working on some **principal amount** of money, denoted by $K$, and the **cap rate** is denoted by $R$.
- The floating rate of interest underlying the cap is not the short rate $r$, but rather some market rate, and we will assume that over the interval $[T_{i-1}, T_i]$ it is the LIBOR spot rate $L(T_{i-1}, T_i)$ (see Section 15.2).
- Caplet $i$ is now defined as the following contingent claim, paid at $T_i$,

$$\chi_i = K\delta \quad \max\left[L(T_{i-1}, T_i) - R, 0\right].$$

We now turn to the problem of pricing the caplet, and without loss of generality we may assume that $K = 1$. We will also use the notation $x^+ = \max[x, 0]$, so the caplet can be written as

$$\chi = \delta(L - R)^+,$$

where $L = L(T_{i-1}, T_i)$. Denoting $p(T_{i-1}, T_i)$ by $p$, and recalling that

$$L = \frac{1 - p}{p\delta},$$

we have

$$\begin{aligned}\chi &= \delta(L - R)^+ = \delta(L - R)^+ = \delta\left(\frac{1 - p}{p\delta} - R\right)^+ \\ &= \left(\frac{1}{p} - (1 + \delta R)\right)^+ = \left(\frac{1}{p} - R^*\right)^+ = \frac{R^*}{p}\left(\frac{1}{R^*} - p\right)^+,\end{aligned}$$

where $R^* = 1 + \delta R$. It is, however, easily seen (why?) that a payment of $\frac{R^*}{p}(\frac{1}{R^*} - p)^+$ at time $T_i$ is equivalent to a payment of $R^*(\frac{1}{R^*} - p)^+$ at time $T_{i-1}$.

Consequently we see that a caplet is equivalent to $R^*$ put options on an underlying $T_i$-bond, where the exercise date of the option is at $T_{i-1}$ and the exercise price is $1/R^*$. An entire cap contract can thus be viewed as a portfolio of put options, and we may use the earlier results of the chapter to compute the theoretical price.

## 19.9 Exercises

**Exercise 19.1** Derive a pricing formula for European bond options in the Ho-Lee model.

**Exercise 19.2 A Gaussian interest rate model**

Take as given an HJM model (under the risk neutral measure $Q$) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma_1 \cdot (T-t)dW_1(t) + \sigma_2 e^{-a(T-t)}dW_2(t)$$

where $\sigma_1$ and $\sigma_2$ are constants.

(a) Derive the bond price dynamics.
(b) Compute the pricing formula for a European call option on an underlying bond.

**Exercise 19.3** Prove that a payment of $\frac{1}{p}(A-p)^+$ at time $T_i$ is equivalent to a payment of $(A-p)^+$ at time $T_{i-1}$, where $p = p(T_{i-1}, T_i)$, and $A$ is a deterministic constant.

**Exercise 19.4** Use the technique above in order to prove the pricing formula of Proposition 17.5 for bond options in the Ho-Lee model.

# 19.10 Notes

The first to use a numeraire different from the risk free asset $B$ were, independently, Geman (1982) and (in a Gaussian framework) Jamshidian (1989), who both used a bond maturing at a fixed time $T$ as numeraire. A systematic study has been carried out by Geman, El Karoui and Rochet in a series of papers, and many of the results above can be found, in a more abstract setting, in Geman *et al.* (1995).

# 20 Forwards and Futures

Consider a financial market of the type presented in the previous chapter, with an underlying factor process $X$, and with a (possibly stochastic) short rate $r$. (If $r$ is stochastic we include it, for brevity of notation, as one component of $X$.) We assume that the market is arbitrage free, and that pricing is done under some fixed risk neutral martingale measure $Q$ (with, as usual, $B$ as numeraire).

Let us now consider a fixed **simple** $T$-claim $\mathcal{Y}$, i.e. a claim of the form $\mathcal{Y} = \Psi(X_T)$, and assume that we are standing at time $t$. If we buy $\mathcal{Y}$ and **pay today**, i.e. at time $t$, then we know that the arbitrage free price is given by

$$\Pi(t, \mathcal{Y}) = G(t, X_t; T, \mathcal{Y}), \tag{20.1}$$

where the pricing function $G$ is given by

$$G(t, x; T, \mathcal{Y}) = E_{t,x}^{Q}\left[\mathcal{Y} \cdot \exp\left\{-\int_t^T r(s)ds\right\}\right], \tag{20.2}$$

and the payment streams are as follows.

1. At time $t$ we pay $\Pi(t, \mathcal{Y})$ to the underwriter of the contract.
2. At time $T$ we receive $\mathcal{Y}$ from the underwriter.

There are two extremely common variations of this type of contract, namely **forwards** and **futures**. Both these contracts have the same claim $\mathcal{Y}$ as their underlying object, but they differ from our standard contract above by the way in which payments are made.

## 20.1 Forward Contracts

We will start with the conceptually easiest contract, which is the forward contract. This is an agreement between two parties to buy or sell a certain underlying claim at a fixed time $T$ in the future. The difference between a forward contract and our standard claims studied so far is that for a forward contract all payments are made at time $T$. To be more precise we give a definition.

**Definition 20.1** *Let $\mathcal{Y}$ be a contingent T-claim. A* **forward contract** *on $\mathcal{Y}$*, **contracted at** $t$, *with* **time of delivery** $T$, *and with the* **forward price** $f(t, T, \mathcal{Y})$, *is defined by the following payment scheme.*

- *The holder of the forward contract receives, at time T, the stochastic amount $\mathcal{Y}$ from the underwriter.*

- *The holder of the contract pays, at time T, the amount $f(t;T,Y)$ to the underwriter.*
- *The forward price $f(t;T,Y)$ is determined at time t.*
- *The forward price $f(t;T,Y)$ is determined in such a way that the price of the forward contract equals zero, at the time t when the contract is made.*

Forward markets are typically not standardized, so forward contracts are usually traded as OTC ("over the counter") instruments. Note that even if the value of a specific forward contract equals zero at the time $t$ of writing the contract, it will typically have a nonzero market value which varies stochastically in the time interval $[t, T]$.

Our immediate project is to give a mathematical formalization of the definition above, and to derive a theoretical expression for the forward price process $f(t;T,Y)$. This turns out to be quite simple, since the forward contract itself is a contingent $T$-claim $\xi$, defined by

$$\xi = Y - f(t;T,Y).$$

We are thus led to the following mathematical definition of the forward price process.

**Definition 20.2** *Let* $Y$ *be a contingent T-claim as above. By the* **forward price process** *we mean a process* $f(t;T,Y)$ *of the form* $f(t;T,Y) = f(t,X_t;T,Y)$, *where f is some determinstic function, with the property that*

$$\Pi(t; Y - f(t,X_t;T,Y)) = 0. \tag{20.3}$$

Writing the forward price as $f(t,X_t;T,Y)$ formalizes the fact that the forward price is determined at $t$, given the information that is available at that time.

We now have the following basic formula for the forward price process.

**Proposition 20.3** *The forward price process is given by any of the following expressions.*

$$f(t;T,Y) = \frac{\Pi(t;Y)}{p(t,T)},$$

$$f(t;T,Y) = f(t,X_t;T,Y), \tag{20.4}$$

*where*

$$f(t,x;T,Y) = \frac{1}{p(t,T)} E^Q_{t,x}\left[ Y \cdot \exp\left\{ -\int_t^T r(s)ds \right\} \right], \tag{20.5}$$

$$f(t,x;T,Y) = E^T_{t,x}[Y]. \tag{20.6}$$

(20.7)

**Proof** Using (20.3) and risk neutral valuation we immediately have the following identities, where we write $f(t, x)$ instead of $f(t,x;T,Y)$

$$\begin{aligned}
0 &= \Pi(t; Y - f(t,X_t)) = E^Q_{t,X_t}\left[ [Y - f(t,X_t)] \cdot \exp\left\{ -\int_t^T r(s)ds \right\} \right] \\
&= E^Q_{t,X_t}\left[ Y \cdot \exp\left\{ -\int_t^T r(s)ds \right\} \right] - E^Q_{t,X_t}\left[ f(t,X_t) \cdot \exp\left\{ -\int_t^T r(s)ds \right\} \right] \\
&= \Pi(t; Y) - f(t,X_t) E^Q_{t,X_T}\left[ \exp\left\{ -\int_t^T r(s)ds \right\} \right] \\
&= \Pi(t; Y) - f(t,X_t)p(t,T).
\end{aligned}$$

This immediately gives us (20.4)–(20.6). The relation (20.7) then follows from (20.5) and Proposition 19.12.

Note that when dealing with forward contracts there is some risk of conceptual confusion. If we fix $t$, $T$ and $\mathcal{Y}$, and let $s$ be a fixed point in time with $t \leq s \leq T$, then there will be two different prices.

1. The forward price $f(s, T, \mathcal{Y})$ which is paid to the underwriter at time $T$ for a forward contract made at time $s$.
2. The (spot) price, at time $s$, of a fixed forward contract, entered at time $t$, and with time $T$ of delivery. This spot price is easily seen to be equal to

$$\Pi(s; \mathcal{Y}) - p(s,T) f(t; T, \mathcal{Y}).$$

## 20.2 Futures Contracts

A futures contract is very much like a forward contract in the sense that it is an agreement between two parties to buy or sell a certain claim at a prespecified time $T$ in the future. The principal difference between the two contracts lies in the way in which payments are being made. We first give a verbal definition of the futures contract.

**Definition 20.4** *Let $\mathcal{Y}$ be a contingent T-claim. A* **futures contract** *on $\mathcal{Y}$, with* **time of delivery** *$T$, is a financial asset with the following properties.*

(i) *At every point of time t with $0 \leq t \leq T$, there exists in the market a quoted object $F(t, T, \mathcal{Y})$, known as the* **futures price** *for $\mathcal{Y}$ at t, for delivery at T.*
(ii) *At the time T of delivery, the holder of the contract pays $F(T; T, \mathcal{Y})$ and receives the claim $\mathcal{Y}$.*
(iii) *During an arbitrary time interval $(s, t]$ the holder of the contract receives the amount $F(t, T, \mathcal{Y}) - F(s; T, \mathcal{Y})$.*
(iv) *The spot price, at any time t prior to delivery, of obtaining the futures contract, is by definition equal to zero.*

A rough way of thinking about a futures contract is to regard it as a forward contract where the payments are made continuously in time in the way described above, rather than all payments being made at time $T$. As the forward price increased, you would then get get richer, and as the forward price decreased you would lose money. The reason that this way of looking at the futures contract is not entirely correct is the fact that if we start with a standard forward contract, with its associated forward price process *f*, and then introduce the above payment scheme over time, this will (through supply and demand) affect the original forward price process, so generically we will expect the futures price process $F$ to be different from the forward price process *f*. The payment schedule above is known as "marking to market"; it is organized in such a way that the holder of a futures position, be it short or long, is required to keep a certain amount of money with the broker as a safety margin against default.

Futures contracts, as opposed to forward contracts, are traded in a standardized manner at an exchange. The volumes in which futures are traded over the world are astronomical, and one of the reasons for this is that on many markets it is difficult to trade (or hedge) directly in the underlying object. A typical example is the commodity market, where you actually have to deliver the traded object (tons of copper, timber, or ripening grapes), and thus are not allowed to go short. In these markets, the futures contract is a convenient financial instrument which does not force you to physically deliver the underlying object, while still making it possible for you to hedge (or speculate) against the underlying object.

We now note some properties of the futures contract.

- From (ii) and (iv) above it is clear that we must have

$$F(T;T,Y) = Y. \tag{20.8}$$

Thus there is really no economic reason to actually deliver either the underlying claim or the payment at time $T$. This is also an empirical fact; the vast majority of all futures contracts are closed before the time of delivery.
- If you enter a futures contract a time $t$ with a corresponding futures price $F(t;T,Y)$, this does **not** mean that you are obliged to deliver $Y$ at time $T$ at the price $F(t;T,Y)$. The only contractual obligation is the payment stream defined above.
- The name futures **price** is therefore somewhat unfortunate from a linguistic point of view. If today's futures price is given by $F(t;T,Y)$ (with $t < T$) this does not mean that anyone will ever pay the amount $F(t;T,Y)$ in order to obtain some asset. It would perhaps be more clear to refer to $F(t;T,Y)$ as the futures **quotation** .
- Since, by definition, the spot price of a futures contract equals zero, there is no cost or gain of entering or closing a futures contract.
- If the reader thinks that a futures contract conceptually is a somewhat complicated object, then the author is inclined to agree.

We now turn to the mathematical formalization of the futures contract, and is should by now be clear that the natural model for a futures contract is an asset with dividends.

**Definition 20.5** *The* **futures contract** *on an underlying T-claim* $Y$ *is a financial asset with a price process* $\Pi(t)$ *and a dividend process* $D(t)$ *satisfying the following conditions.*

$$D(t) = F(t, T, \mathrm{Y}).$$

$$F(T, T, \mathrm{Y}) = \mathrm{Y}. \tag{20.9}$$

$$\Pi(t) = 0, \quad \forall t \leq T. \tag{20.10}$$

(20.11)

It now remains to investigate what the futures price process looks like. This turns out to be quite simple, and we can now prove the main result of the section.

**Proposition 20.6** *Let* Y *be a given contingent T-claim, and assume that market prices are obtained from the fixed risk neutral martingale measure Q. Then the following hold.*

- *The futures price process is given by*

$$F(t, T, \mathrm{Y}) = E^{Q}_{t,X_t}[\mathrm{Y}]. \tag{20.12}$$

- *If the short rate is deterministic, then the forward and the futures price processes coincide, and we have*

$$f(t, T, \mathrm{Y}) = F(t, T, \mathrm{Y}) = E^{Q}_{t,X_t}[\mathrm{Y}]. \tag{20.13}$$

**Proof** From Proposition 11.14, it follows that the discounted gains process

$$G^{Z}(t) = \frac{\Pi(t)}{B_t} + \int_0^t \frac{1}{B_s} dF(s, T, \mathrm{Y})$$

has the representation

$$dG^{Z}_t = h_t dW_t$$

for some adapted process *h*. In our case we furthermore have $\Pi(t)=0$ for all *t*, so we obtain the representation

$$\frac{1}{B_t} dF(t, T, \mathrm{Y}) = h_t dW_t.$$

Multiplying by $B_t$ on both sides we get

$$dF(t, T, \mathrm{Y}) = B_t h_t dW_t,$$

which implies that $F(t, T, \mathrm{Y})$ is a *Q*-martingale. Using the martingale property, the fact that we are in a Markovian framework (see Lemma 4.9), and (20.10), we obtain

$$F(t, T, \mathrm{Y}) = E^{Q}_{t,X_t}[F(T, T, \mathrm{Y})] = E^{Q}_{t,X_t}[Y],$$

which proves the first part of the assertion. The second part follows from the fact that $Q^T=Q$ when the short rate is deterministic (see Lemma 19.11).

## 20.3 Exercises

**Exercise 20.1** Suppose that $S$ is the price process of a non dividend paying asset. Show that the forward price $f(t,x;T,Y)$ for the $T$-claim $Y = S_T$ is given by

$$f(t,x;T,S_T) = \frac{S_t}{p(t,T)}.$$

**Exercise 20.2** Suppose that $S$ is the price process of a dividend paying asset with dividend process $D$.

(a) Show that the forward price $f(t,x;T,S_T)$ is given by the cost of carry formula

$$f(t,x;T,S_T) = \frac{1}{p(t,T)}\left(S_t - E^Q_{t,x}\left[\int_t^T \exp\left\{-\int_t^s r(u)du\right\}dD_s\right]\right).$$

**Hint:** Use the cost of carry formula for dividend paying assets.

(b) Now assume that the short rate $r$ is deterministic but possibly time-varying. Show that in this case the formula above can be written as

$$f(t,x;T,S_T) = \frac{S_t}{p(t,T)} - E^Q_{t,x}\left[\int_t^T \exp\left\{-\int_s^T r(u)du\right\}dD_s\right].$$

**Exercise 20.3** Suppose that $S$ is the price process of an asset in a standard Black--Scholes model, with $r$ as the constant rate of interest, and fix a contingent $T$-claim $\Phi(S(T))$. We know that this claim can be replicated by a portfolio based on the money account $B$, and on the underlying asset $S$. Show that it is also possible to find a replicating portfolio, based on the money account and on futures contracts for $S(T)$.

## 20.4 Notes

For a wealth of information on forwards and futures, see Hull (1997) and Duffie (1989).

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